SCHUR INDICES OF PROJECTIVE REPRESENTATIONS OF HYPEROCTAHEDRAL GROUPS

By

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This thesis is dedicated to my parents.

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The hyperoctahedral group, denoted by W_n, is the Weyl group of the root systems B_n and C_n, or equivalently, the symmetry group of an n-dimensional cube.

The purpose of this thesis is to study the Schur Indices of the projective representations of W_n. This job, by Schur's result, can be reduced to studying the ordinary representations of the representation groups of W_n. As is well known, the Schur indices of all ordinary irreducible representations of W_n are always one. In fact, there is a general result about the Schur index of Weyl groups of simple Lie algebras, which says that the Schur indices of all ordinary irreducible representations of the Weyl groups of Lie types are always one. Because of this, the Schur Indices of the projective representations of W_n can be further reduced to studying the spin representations of the double covers of W_n. Our work is divided into four parts.

We first start by studying the structures of the double covers of W_n and we show that there are 32 distinct double covers (up to isomorphism) and these 32

double covers fall into 8 families of spin representations. The conjugacy classes of these double covers have to be studied in detail.

Next, we develop some tools in order to study the spin representations. This includes introducing A. Turull's group pair technique, Stembridge's $\mathbf{Z}_2$-quotient technique, and the Brauer-Wall group technique.

Then, we define and construct some basic spin representations including $\Theta^{\epsilon_1,\epsilon_2}$, $\Phi^{\lambda,\mu}$, $\rho^{\epsilon_1,\epsilon_2}$ and $\Psi^{\epsilon_1,\epsilon_2}$. We also quote some results about ordinary representations $X^{\lambda,\mu}$ of W_n and projective representations φ^λ of Symmetric groups S_n.

Finally, after we construct all the spin representations of W_n in terms of $\Theta^{\epsilon_1,\epsilon_2}$, $\Phi^{\lambda,\mu}$, $\rho^{\epsilon_1,\epsilon_2}$, $\Psi^{\epsilon_1,\epsilon_2}$ and $X^{\lambda,\mu}$, for each irreducible spin representation χ of W_n, we give a complete calculation of the division algebra associated with χ. This is our final goal.

Our results enable us to calculate the exact Schur index over $\mathbf{Q}$ for each spin representation of the 32 double covers of W_n.

Thus, the exact calculation of Schur indices of projective representations of Hyperoctahedral Group is completely solved.

CHAPTER 1
INTRODUCTION

The Hyperoctahedral Group W_n is the Weyl group of the root systems B_n and C_n, or equivalently, the symmetry group of an n-dimensional cube. For $n \geq 4$, this group has a Schur multiplier isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$. This implies, in particular, that the irreducible projective representations of W_n (over the complex field $\mathbf{C}$) can be viewed as certain ordinary representations of some double covers of W_n.

If χ is an irreducible character of some finite group G and $\mathbf{K}$ is a field of characteristic zero, the Schur index $m_{\mathbf{K}}(\chi)$ of χ with respect to $\mathbf{K}$ is the least positive multiplicity of χ in a character afforded by a $\mathbf{K}G$-module. The question is: what are the Schur indices of the characters of the representation groups of W_n? We give here a method that will tell us how to describe the Schur index of each irreducible character of the representation groups (or equivalently, the double covers) of W_n, for all n.

A substantial amount of research has been devoted to the Schur index. Efforts have been directed both to finding its general properties and to calculating it for important classes of groups. For example, Bernard [1] showed that the Schur index of every character of the Weyl groups of type E_6, E_7 and E_8 is one. Feit [2] gives short proofs for the calculation of the Schur index for many specific groups, including Bernard's Weyl groups of types E_6, E_7 and E_8, and the representation groups of Sporadic Simple Groups. Other authors have studied the Schur index for groups of Lie type, see for example Gow [4] or Morris [13]. Various methods are available to give upper bounds for the Schur indices, but it is usually harder to find their value when they are larger than one. Hence, for the groups of Lie type, one can show that

their Schur indices are at most two in many cases and at most one in some cases, but the question of exactly for what characters it is one and for which it is two has not been consistently answered.

Relatively few results have been obtained, since Schur's paper [20], on the character theory of the representations of S_n and A_n. Schur himself calculated the Schur indices of these characters over the reals in a paper published in 1927. Some new constructions have been found for the characters; see for example Morris [13], Read [18]. A Hopf algebra approach along the lines of Zelevinski [30] has been taken and some basic facts and combinatorial results on the characters of the representation groups of S_n and A_n have been proved in this context; see for example Hoffman [5]. The combinatorial theory of shifted tableaux has been used by Stembridge [22] to give new proofs of Schur's basic results and to obtain a shifted analogue of the Littlewood-Richardson rule as well as some other combinatorial results. The construction of the irreducible projective representations of S_n over $\mathbf{C}$ has recently been obtained by Nazarov in [15].

It is a well-known result that the Schur index of the ordinary characters of the symmetric and alternating groups is always one. Turull [24] has shown that the Schur index of the characters corresponding to the exceptional covers of A_n (for $n = 6, 7$) is always one over $\mathbf{Q}$. For the characters of the double covers of S_n and A_n, a simple argument given by Turull [24] shows that the Schur index is always one or two. Turull also gives, in another paper [25], a simple combinatorial rule to calculate the Schur index for all such characters.

It is also well-known that the Schur index of the ordinary characters of the group W_n is always one over $\mathbf{Q}$ (see Young [29]). Zelevinski in [30] gave a new approach to the same problem by using Hopf algebra. In fact, there is a general result about the Schur index of the ordinary irreducible representations of the Weyl groups of Lie type, which says that the Schur index of the ordinary representations

of the Weyl group of Lie type is always one over $\mathbf{Q}$. Readers may refer to Bernard [1], Kondo [11], Specht [21] and Young [29] for the details.

Because the Schur index of the ordinary characters of the group W_n is always one over $\mathbf{Q}$, the study of Schur index of projective representation is reduced to the study of the Schur index of the spin representations of the double covers of W_n.

Stembridge [23] considered the hyperoctahedral groups W_n and gave construction of all irreducible representations and characters of 32 double covers of W_n over the complex $\mathbf{C}$. As a corollary, he also obtained the irreducible projective representations and characters of the Weyl group of the root system D_n.

Turull's technique [25] and Stembridge's results and methods [23] enable us to calculate the Schur indices of the projective characters of W_n.

As is well-known, the Schur index is the rank of a division algebra associated with each character (The details will be studied in Chapter Four). The purpose of this dissertation is to calculate this algebra for each character.

This dissertation is organized as follows. It is divided into four Chapters. Chapter One: Introduction, Chapter Two: Preliminaries, Chapter Three: Representations and Chapter Four: Schur Indices.

Chapter Two is the foundation of the dissertation. We begin with studying the double covers of W_n and their conjugacy classes. Then we study the Clifford theory for $\mathbf{Z}_2^2$-quotients, and then study the quaternion algebras and Brauer-Wall groups. Finally, we study the relations between Schur index and group pairs and some other useful results are established for later use.

In Chapter Three, we define and construct some basic spin representations including $X^{\lambda,\mu}$, $\Theta^{\varepsilon_1,\varepsilon_2}$, $\Phi^{\lambda,\mu}$, $\rho^{\varepsilon_1,\varepsilon_2}$, $\Psi^{\varepsilon_1,\varepsilon_2}$. We also quote some results from the ordinary representations of W_n and the projective representations of symmetric groups. In Chapter Four we show that all the spin representations of W_n can be constructed from these basic spin representations.

In Chapter Four, we turn to the problem of calculating the division algebras associated with characters for each of the eight families of double covers. This task can be roughly divided into two phases. In the first phase, we construct the modules and describe the characters for each of the eight families of double covers, which follows Stembridge's idea [23]. In the second phase, we use group pair technique to calculate the division algebras for each character, which follows Turull's idea [25].

CHAPTER 2
PRELIMINARIES

This chapter is the foundation of the dissertation. We begin with studying the double covers of W_n and their conjugacy classes. Then we study the Clifford theory for $\mathbf{Z}_2^2$-quotients and introduce the group pair technique. Then we study the quaternion algebras and Brauer-Wall groups. Finally, we recall some fundamental concepts about Schur indices and study the relations between Schur index and group pairs and some other useful results are established for later use.

2.1 The Double Covers of W_n

Let $s_1, s_2, ..., s_{n-1}, t$ denote a set of Coxeter generators of W_n. We will assume that these generators are labeled so that for the reflection representation, s_i corresponds to interchanging coordinates i and $i+1$, and t corresponds to changing the sign of the first coordinate. Note that the Coxeter relations

$$s_i^2 = t^2 = (s_i s_{i+1})^3 = 1,$$

$$[s_i, s_j] = 1 \ (\ |i-j| \geq 2),$$

$$[s_i, t] = 1 \ (i > 1), \ [s_1 t s_1^{-1}, t] = 1$$

constitute a presentation of W_n.

Let $L_n = Hom(W_n, \mathbf{C}^\times)$ denote the abelian group of linear characters of W_n. An easy application of the Coxeter relations shows that L_n is generated by the two characters ε and δ defined by

$$\varepsilon(s_i) = -1, \quad \varepsilon(t) = +1,$$

$$\delta(s_i) = +1, \quad \delta(t) = -1.$$

This shows in particular that $L_n \simeq \mathbf{Z}_2 \times \mathbf{Z}_2$. We summarize this in the following lemma.

Lemma 2.1.1 *$W_n/W_n' \simeq \mathbf{Z}_2 \times \mathbf{Z}_2$, W_n has four linear characters. Let*

$$\varepsilon(s_i) = -1, \quad \varepsilon(t) = +1,$$
$$\delta(s_i) = +1, \quad \delta(t) = -1.$$

then the four linear characters are

$$1, \varepsilon, \delta, \epsilon\delta.$$

To study the projective representations of W_n over $\mathbf{C}$, by Schur's theory, we can equivalently study the ordinary representations of the representation groups of W_n. Since the Schur multiplier of W_n is isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$ for $n \geq 4$, $\mathbf{Z}_2 \times \mathbf{Z}_2$ for $n = 3$ and $\mathbf{Z}_2$ for $n = 2$ (cf. Ihard and Yokonuma [8], Howlet [7]), we can further reduce our problem to studying the ordinary representations of double covers $\tilde{W}_n$ of W_n.

Actually, we are only interested in the complex representations of $\tilde{W}_n$ for which (-1) is sent to -Id. These representations are called Zweiter Art by Schur [20] and they are sometimes also called spin representations. To be consistent, we call them the spin representations and spin characters in this dissertation.

For each double cover $\tilde{W}_n$ of W_n, we have the following extension:

$$1 \to \{-1, 1\} \to \tilde{W}_n \to W_n \to 1 \tag{2.1}$$

with $\{-1, 1\} \subseteq Z(\tilde{W}_n)$.

Next proposition classifies all double covers of W_n.

Proposition 2.1.2 *There are 32 double covers of W_n and their generators σ_i, τ have the following presentations of the form*

$$\sigma_i^2 = \varepsilon_1, \; \tau^2 = \varepsilon_2, \; (\sigma_i\sigma_{i+1})^3 = \varepsilon_1,$$

$$[\sigma_i, \sigma_j] = \varepsilon_3 \ (|i-j| \geq 2),$$

$$[\sigma_i, \tau] = \varepsilon_4 \ (i > 1), \ [\sigma_1 \tau \sigma_1^{-1}, \tau] = \varepsilon_5, \tag{2.2}$$

for suitable $\varepsilon_1, \ \varepsilon_2, \ \varepsilon_3, \ \varepsilon_4, \ \varepsilon_5 = \pm 1$.

Proof: Let σ_i and τ denote a preimage of s_i and t in (2.1), respectively. It follows from the presentation of W_n that

$$\sigma_i^2, \tau^2 \in \{-1, 1\},$$

$$(\sigma_i \sigma_{i+1})^3 \in \{-1, 1\},$$

$$[\sigma_i, \sigma_j] \in \{-1, 1\} \ (|i-j| \geq 2),$$

$$[\sigma_i, \tau] \in \{-1, 1\} \ (i \geq 1),$$

$$[\sigma_1 \tau \sigma_1^{-1}, \tau] \in \{-1, 1\}.$$

Let $\varepsilon_2 \in \{-1, 1\}$ such that $\tau^2 = \varepsilon_2$.

Let $\tilde{W}_n$ be the double cover of W_n generated by $\sigma_1, \sigma_2, \cdots, \sigma_{n-1}, \tau, (-1)$. If the images of x, y of $\tilde{W}_n$ are conjugate in W_n, then there exists an element $z \in \tilde{W}_n$ such that $zxz^{-1} = \pm y$. In particular, this shows that x^2 and y^2 are $\tilde{W}_n$-conjugates. Therefore, since the involutions s_i $(i = 1, 2, ..., n-1)$ are all W_n-conjugates, then the σ_i^2 are all $\tilde{W}_n$-conjugates. In other words, there exists an element $\varepsilon_1 \in \{-1, 1\}$ independent of i such that $\sigma_i^2 = \varepsilon_1$. Similarly, since the involutions $s_i s_j$ $(|i-j| \geq 2)$ are all W_n-conjugates, then $(\sigma_i \sigma_j)^2$ $(|i-j| \geq 2)$ are all $\tilde{W}_n$-conjugates. Since $(\sigma_i \sigma_j)^2 = [\sigma_i, \sigma_j]\varepsilon_1 \varepsilon_2$, we conclude that $[\sigma_i, \sigma_j]$ are all $\tilde{W}_n$-conjugates. Therefore, there exists an element $\varepsilon_3 = \pm 1$ independent of i and j such that $[\sigma_i, \sigma_j] = \varepsilon_3$. By the same argument we can show that there exist elements ε_4 and ε_5 such that $\varepsilon_4, \varepsilon_5 \in \{-1, 1\}$, $[\sigma_i, \tau] = \varepsilon_4$ $(i > 1)$ and $[\sigma_1 \tau \sigma_1^{-1}, \tau] = \varepsilon_5$.

Finally, we may substitute $\sigma_i \longmapsto -\sigma_i$, if necessary, to ensure that $(\sigma_i \sigma_{i+1})^3 = \varepsilon_1$. Then the presentations follow.

Each double cover of W_n is associated with five parameters $\alpha = [\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5]$. We call α the factor set of W_n. It is easy to check that for different factor sets, the corresponding double covers are not isomorphic. Therefore, there are 32 double covers of W_n (up to isomorphism). This finishes the proof of the proposition. ■

From now on, for each factor set $\alpha = [\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5]$, we use the notation $W_n[\alpha]$ to denote the double cover associated with the factor set α. Sometimes we can also explicitly write $W_n[\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5]$ for the double cover $W_n[\alpha]$.

As Stembridge [23] has shown, it is convenient to record a few basic identities that will be needed later. To simplify the notation let us define

$$t_i = s_{i-1} \cdots s_1 t s_1 \cdots s_{i-1} \in W_n,$$

$$\tau_i = \sigma_{i-1}^{-1} \cdots \sigma_1^{-1} \tau \sigma_1 \cdots \sigma_{i-1} \in W_n[\alpha] \tag{2.3}$$

for $1 \le i \le n$.

Note that $t_1, t_2, \ldots, t_n$ are the reflections corresponding to the short roots of the root system B_n; we will refer to them as the "short reflections".

In the following, $[x, y] = xyx^{-1}y^{-1}$, as above, denotes the group commutator in $W_n[\alpha]$.

Proposition 2.1.3 *Assuming* $\alpha = [\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5]$, *we have*

$$[\sigma_i, \sigma_j] = \varepsilon_3 \ (|i - j| \ge 2),$$

$$[\sigma_i, \tau_j] = \varepsilon_4 \ (j - i \ne 0, 1),$$

$$[\tau_i, \tau_j] = \varepsilon_5 \ (i \ne j).$$

Proof: The first relation is just one of (2.1.2). The proofs of the second and the third relations are very straightforward. ■

For factor sets $\alpha = [\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5,]$ and $\beta = [\varepsilon'_1, \varepsilon'_2, \varepsilon'_3, \varepsilon'_4, \varepsilon'_5]$, we can define their product

$$\alpha\beta = [\varepsilon_1\varepsilon'_1, \varepsilon_2\varepsilon'_2, \varepsilon_3\varepsilon'_3, \varepsilon_4\varepsilon'_4, \varepsilon_5\varepsilon'_5].$$

The product $\alpha\beta$ is also a factor set of W_n and there is a double cover $W_n[\alpha\beta]$ associated with $\alpha\beta$.

In the rest of this section, we will develop the relations between three groups $W_n[\alpha]$, $W_n[\beta]$ and $W_n[\alpha\beta]$ in terms of their modules. To do that, we will take another form of presentation of double covers of W_n.

First, we give the following definition.

Definition 2.1.4 *Let A be a possibly infinite abelian group and let G be any group. Then an A-2-cocycle of G is a function $f : G \times G \to A$ such that*

$$f(xy, z)f(x, y) = f(x, yz)f(y, z)$$

for all $x, y, z \in G$.

The 2-cocycle has close relation with the theory of group extensions. Let $W_n[\alpha]$ be a double cover of W_n associated with factor set $\alpha = [\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5]$. Recall (2.1)

$$1 \to \{-1, 1\} \to W_n[\alpha] \to W_n \to 1$$

with $\{-1, 1\} \subseteq Z(W_n[\alpha])$.

As is well known from the theory of group extensions, $W_n[\alpha]$ is determined by the equivalence class of a 2-cocycle

$$f : W_n \times W_n \to \{-1, 1\}.$$

Because of this, sometimes, we write

$$f = [\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5]$$

and denote $W_n[\alpha]$ by $W_n[f]$. Also we call the 2-cocycle f a factor set of W_n. So in the rest of dissertation, we will use both concepts. They are equivalent in the sense that they determine (up to isomorphism) and are determined by the double cover.

For each 2-cocycle f of W_n, we define $W_n[f]$ to be the set $\{-1,1\} \times W_n$ and the multiplication is given by

$$(\nu_1, w_1)(\nu_2, w_2) = (\nu_1\nu_2 f(w_1, w_2), w_1 w_2) \tag{2.4}$$

where $\nu_i \in \{-1,1\}$ and $w_i \in W_n$ $(i = 1,2)$.

It is routine to check that (2.4) does define a group multiplication.

Remark : With the new form of presentation of $W_n[f]$, the element $(1, s_i)$ corresponds to σ_i and the element $(1,t)$ corresponds to τ.

For each 2-cocycle f of W_n, there is a double cover $W_n[f]$ associated with it. Let α, β be two factor sets of W_n and f and f' be their corresponding 2-cocycles. It is easy to check that the product $\alpha\beta$ corresponds to the product ff'. Now we are ready to state the following theorem.

Theorem 2.1.5 *Let f and f' be two 2-cocycles of W_n. Let $\mathbf{K}$ be a field of characteristic zero. Let V_1 be a $\mathbf{K}W_n[f]$-module affording the spin representation Φ, let V_2 be a $\mathbf{K}W_n[f']$-module affording the spin representation Ψ, then the tensor product $V_1 \otimes V_2$ affords a spin representation of $W_n[ff']$, denoted by $\Phi\Psi$, via the following action:*

$$(\nu, w)(v_1 \otimes v_2) = \nu\left((1,w)v_1 \otimes (1,w)v_2\right) \tag{2.5}$$

where (ν, w) denotes a typical element of $W_n[ff']$, i.e., $\nu \in \{-1,1\}$ and $w \in W_n$.

Proof: It is straightforward to check that (2.5) does define a $W_n[ff']$ action on the tensor product $M \otimes N$ and $\Phi\Psi$ is a spin representation, i.e., $\Phi\Psi(-1) = -1$. ∎

Note that the notation $\Phi\Psi$ is not the regular product of two representations. The regular product of two representations involves two representations from the same group, but here the two representations come from two different groups. Since we will use this kind of product quite often, it is better to give a name to it. Let's call $\Phi\Psi$ the twisted product of Φ and Ψ. This convention will be also applied to

their characters. If φ and ψ are the characters afforded by Φ and Ψ, then we use the notation $\varphi\psi$ to denote the character afforded by $M \otimes N$ as a $W_n[\alpha\beta]$-module, and call $\varphi\psi$ the twisted product of φ and ψ. The tensor product $M \otimes N$ with the group action defined in this way is called the twisted tensor product of M and N. We will use this convention without further mention in the rest of dissertation unless otherwise particularly stated.

Another thing I would like to point out is that the regular product of two spin representations of the same group is no longer a spin representation, but the twisted product we just defined in (2.5) is a spin representation.

Remarks: Given a spin representation of the group $W_n[\alpha]$ and a spin representation of group $W_n[\beta]$, the above construction allows us to obtain a new spin representation of the group $W_n[\alpha\beta]$. This is a main technique we use in Chapter Four. This enables us to create many new spin representations from given spin representations of groups $W_n[\alpha]$ and $W_n[\beta]$.

Lemma 2.1.6 *If φ is a linear spin representation of $W_n[\alpha]$ and χ is an irreducible representation of $W_n[\beta]$, then the twisted product $\varphi\chi$ is an irreducible representation of $W_n[\alpha\beta]$.*

Proof: We show $<\varphi\chi, \varphi\chi>= 1$. In fact

$$\begin{aligned}
<\varphi\chi, \varphi\chi> &= \frac{1}{2|W_n|} \sum_{\nu\in\{-1,1\}, w\in W_n} \varphi\chi(\nu, w)\overline{\varphi\chi(\nu, w)} \\
&= \frac{1}{2|W_n|} \sum_{\nu\in\{-1,1\}, w\in W_n} \nu\varphi(1, w)\chi(1, w)\overline{\nu\varphi(1, w)\chi(1, w)} \\
&= \frac{1}{2|W_n|} \sum_{\nu\in\{-1,1\}, w\in W_n} \varphi(\nu, w)\chi(1, w)\overline{\varphi(\nu, w)\chi(1, w)} \\
&= \frac{1}{2|W_n|} \sum_{\nu\in\{-1,1\}, w\in W_n} \chi(1, w)\overline{\chi(1, w)} \\
&= \frac{1}{2|W_n|} \sum_{\nu\in\{-1,1\}, w\in W_n} \nu\chi(1, w)\overline{\nu\chi(1, w)}
\end{aligned}$$

$$= \frac{1}{2|W_n|} \sum_{\nu\in\{-1,1\},w\in W_n} \chi(\nu,w)\overline{\chi(\nu,w)}$$
$$= <\chi,\chi>$$
$$= 1.$$

The proof is complete. ■

2.2 The Twisted Algebras and Double Covers of W_n

When studying the projective representations of W_n, some authors, for example Stembridge [22] and [23], use two different approaches, one is the twisted algebra of W_n, the other is the double covers of W_n. These two approaches are equivalent when studying the projective representations of W_n. In this section, we will clarify this fact and establish the relations between the twisted algebras and double covers of W_n.

First we give the following definition of projection representation.

Definition 2.2.1 *Let G be a group and $\mathbf{K}$ a field. Let $\mathcal{X} : G \to GL(n, \mathbf{K})$ be such that for every $g, h \in G$, there exists a scalar $f(g, h) \in \mathbf{K}^\times$ such that*

$$\mathcal{X}(g)\mathcal{X}(h) = \mathcal{X}(gh)f(g,h).$$

We call $\mathcal{X}$ a projective $\mathbf{K}$-representation of G. Its degree is n and the function $f : G \times G \to \mathbf{F}^\times$ is called the 2-cocycle of G associated with $\mathcal{X}$.

That f is really a 2-cocycle of W_n follows from the associative property of multiplication of W_n.

Thus, the 2-cocycle associated with a projective $\mathbf{K}$-representation is an $\mathbf{K}^\times$-2-cocycle where $\mathbf{K}^\times$ denotes the multiplicative group of $\mathbf{K}$. Conversely, every $\mathbf{K}^\times$-2-cocycle is associated with a projective $\mathbf{K}$-representation. To see this, we introduce the twisted group algebra as follows.

Let G be a finite group and $\mathbf{K}$ a field. Let f be an $\mathbf{K}^\times$-2-cocycle of G. Let $\mathbf{K}^f[G]$ be the $\mathbf{K}$-vector space with basis $\{c_g | g \in G\}$. (That is, there is a specific basis

of $\mathbf{K}G^f$ which is in one-to-one correspondence with G.) Define multiplication in $\mathbf{K}G^f$ by $c_g c_h = c_{gh} f(g,h)$ and extend via the distributive law. To establish that the multiplication thus defined is associative, it suffices to check it on the basis elements c_g for $g \in G$. That it holds then is immediate from the definition of a 2-cocycle. The finite dimensional algebra $\mathbf{K}G^f$ is the twisted group algebra with respect to f. Note that if f is the trivial $\mathbf{K}^\times$-2-cocycle, that is, $f(g,h) = 1$, for all g, h, we can identify $\mathbf{K}G^f$ with $\mathbf{K}G$.

Now let f be an $\mathbf{K}^\times$-2-cocycle of G and let $\mathcal{Y}$ be any representation of the algebra $\mathbf{K}G^f$. Define $\mathcal{X}(g) = \mathcal{Y}(c_g)$. Then $\mathcal{X}$ is nonsingular and

$$\begin{aligned} \mathcal{X}(g)\mathcal{X}(h) &= \mathcal{Y}(c_g)\mathcal{Y}(c_h) \\ &= \mathcal{Y}(c_g c_h) \\ &= \mathcal{Y}\left(c_{gh} f(g,h)\right) \\ &= \mathcal{X}(gh) f(g,h), \end{aligned}$$

so that $\mathcal{X}$ is a projective representation of G with factor set f.

Conversely, if $\mathcal{X}$ is a projective $\mathbf{K}$-representation of G with 2-cocycle f, we can define a representation $\mathcal{Y}$ of $\mathbf{K}G^f$ by setting $\mathcal{Y}(c_g) = \mathcal{X}(g)$ and extending by linearity. In other words, the projective $\mathbf{K}$-representations of G having 2-cocycle f are in a natural one-to-one correspondence with the representations of the twisted group algebra $\mathbf{K}G^f$. The situation is analogous to the connection between ordinary representations and the ordinary group algebra.

The following fact is well-known.

Lemma 2.2.2 *Let f be an $\mathbf{K}^\times$-2-cocycle of G. Then G has irreducible projective $\mathbf{K}$-representations with 2-cocycle f.*

From the above discussion, we know that the study of representations of the twisted algebra of W_n is equivalent to the study of projective representations of

W_n. On the other hand, we also mentioned earlier that the study of projective representations of W_n is equivalent to the study of the ordinary representations of the double covers of W_n. This implies that the study of the representations of the twisted algebra $\mathbf{K}W_n^f$ is equivalent to the study of the representations of the double covers $W_n[f]$, where f is a 2-cocyle of W_n such that

$$f : W_n \to \{1, -1\}.$$

The remainder of this section will clarify this fact in some detail.

For each 2-cocycle f of W_n, and a field $\mathbf{K}$ of characteristic zero, there are a double cover $W_n[f]$ and a twisted algebra $\mathbf{K}W_n^f$ associated with f. To establish the relations between $W_n[f]$ and $\mathbf{K}W_n^f$, and also to be consistent, we rewrite the twisted algebra $\mathbf{K}W_n^f$ as the set of formal sums

$$\{\sum_{w \in W_n} (x_w, w) | x_w \in \mathbf{K}\}$$

under the addition

$$\sum_{w \in W_n} (x_w, w) + \sum_{w \in W_n} (y_w, w) = \sum_{w \in W_n} (x_w + y_w, w).$$

The multiplication is defined to be

$$(x_1, w_1)(x_2, w_2) = (x_1 x_2 f(w_1 w_2), w_1 w_2)$$

and extends by the distributive property. The scalar action is defined via

$$k \sum_{w \in W_n} (x_w, w) = \sum_{w \in W_n} (k x_w, w).$$

Obviously from the above definitions, the twisted algebra $\mathbf{K}W_n^f$ contains a subset

$$\{(\nu, w) | \nu \in \{-1, 1\}, w \in W_n\}$$

which forms a group under multiplication and is isomorphic to $W_n[f]$. Because of this, we have the following theorem.

Theorem 2.2.3 *There is a one-to-one correspondence between the spin representations of $W_n[f]$ and the representations of the twisted group algebra $\mathbf{K}W_n^f$.*

Proof: The proof is straightforward. ∎

We finish this section by stating a result similar to Lemma 2.1.1. Recall (2.1)

$$1 \to \{-1, 1\} \to W_n[\alpha] \to W_n \to 1.$$

The four linear characters $1, \varepsilon, \delta, \varepsilon\delta$ can be viewed as linear characters of the double cover $W_n[\alpha]$. Therefore, we have the following lemma.

Lemma 2.2.4 *For any factor set α, the group $W_n[\alpha]$ has 4 linear characters*

$$1, \varepsilon, \delta, \varepsilon\delta$$

which send -1 to 1, such that

$$\varepsilon(\nu, w) = \varepsilon(w),$$
$$\delta(\nu, w) = \delta(w)$$

where (ν, w) denotes the standard element of the double cover $W_n[\alpha]$, namely, $\nu \in \{-1, 1\}, w \in W_n$.

Remark : In the above lemma, the left-side ε denotes the linear character of group $W_n[\alpha]$ and the right-side ε denotes the linear character of group W_n. A similar explanation is applied to the notation δ. Even though we use the same notation, it is easy to understand from the context.

The notations $1, \varepsilon, \delta, \varepsilon\delta$ for the four linear characters will maintain the above meanings and be used throughout the dissertation unless otherwise mentioned. The $1, \varepsilon, \delta, \varepsilon\delta$ are all the linear characters of $W_n[\alpha]$ except when $\alpha = [\varepsilon_1, \varepsilon_2, 1, 1, 1]$, in that situation, the group $W_n[\alpha]$ has 8 linear characters (see Section 3.4).

2.3 Conjugacy Classes

In this section, we will classify the conjugacy classes of each group $W_n[\alpha]$ in order to simplify the task of describing the irreducible characters.

We will use P to denote the set of partitions; i.e., sequences of zero or more positive integers of the form $\lambda = (\lambda_1 \geq \ldots \geq \lambda_l)$. We will use $|\lambda|$ to denote the sum of the parts, and $l(\lambda)$ to denote the number of parts. The modifiers E, O and D will be used to indicate that the parts should be restricted to be even, odd, or distinct, respectively. For example, the notation DOP will thus refer to the set of partitions with distinct, odd parts. Also, we will use DP^+ (rep., DP^-) to refer to the partitions $\lambda \in DP$ for which $|\lambda| - l(\lambda)$ is even (rep., odd).

First, consider the conjugacy classes of W_n. These are indexed by ordered pairs of partitions (λ, μ) with $|\lambda| + |\mu| = n$. To describe the members of a given class, let us identify W_n with the group of $n \times n$ monomial matrices with entries chosen from $\{0, 1, -1\}$. References to the cycles of an element $\omega \in W_n$ will thus refer to the cycles of the underlying permutation matrix. We will say that a cycle of ω is positive or negative according to whether the number of -1's in the matrix entries of the cycle is even or odd. In these terms, the class indexed by (λ, μ) consists of those $\omega \in W_n$ whose positive (respectively, negative) cycles have length $\lambda_1, \lambda_2, \ldots$ (respectively, $\mu_1, \mu_2, \ldots$). We remark that the values of the linear characters ε and δ on the (λ, μ)-class are $(-1)^{n-l(\lambda)-l(\mu)}$ and $(-1)^{l(\mu)}$, respectively. For further details, see James and Kerber [10] or Section 7 of Zelevinski [30].

Now, choose a particular W_n-factor set $\alpha = [\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5]$ and let $C(\alpha)$ denote the inverse image in $W_n[\alpha]$ of some W_n-class C. If any $\omega \in C(\alpha)$ is conjugate to $-\omega$, then $C(\alpha)$ is a $W_n[\alpha]$-class; otherwise, $C(\alpha)$ splits into two such classes. Thus, the essential structure of the $W_n[\alpha]$-classes can be inferred from the knowledge of which pairs (λ, μ) index W_n-classes that split in $W_n[\alpha]$.

Theorem 2.3.1 *For each factor set α, the pairs of partitions (λ, μ) that index split classes of $W_n[\alpha]$ can be found in Table I. The results do not depend on the first two parameters ε_1 and ε_2. So in the Table we did not include the first two parameters for each factor set.*

The lists in Table I have been broken up into four columns according to the four possible values of ε and δ on a given W_n-class. For example, the entry in the fifth column for the factor set $[\varepsilon_1, \varepsilon_2, 1, -1, -1]$ is (DOP, DEP). This means that if (λ, μ) indexes a W_n-class with $\lambda = \delta = -1$, then the inverse image of this class splits in $W_n[\varepsilon_1, \varepsilon_2, 1, -1, -1]$ if and only if the parts of λ are distinct and odd and those of μ are distinct and even.

Table I

α	$\varepsilon = +1, \delta = +1$	$\varepsilon = -1, \delta = +1$	$\varepsilon = +1, \delta = -1$	$\varepsilon = -1, \delta = -1$
$[+1, +1, +1]$	(P, P)	(P, P)	(P, P)	(P, P)
$[+1, -1, +1]$	(P, P)	$(EP, \emptyset)$	(DOP, DOP)	$(\emptyset, EP)$
$[-1, +1, +1]$	(OP, OP)	(DP, DP)	(OP, OP)	(DP, DP)
$[-1, -1, +1]$	(OP, OP)	$(DEP, \emptyset)$	(DP, DP)	$(\emptyset, DEP)$
$[+1, +1, -1]$	$(OP, \emptyset)$	$\emptyset$	$(\emptyset, DP)$	$(\emptyset, DP)$
$[+1, -1, -1]$	$(OP, \emptyset)$	$(\emptyset, DP)$	$(\emptyset, OP)$	(DOP, DEP)
$[-1, +1, -1]$	(OP, EP)	$\emptyset$	$(\emptyset, DOP)$	$(\emptyset, P)$
$[-1, -1, -1]$	(OP, EP)	$(\emptyset, P)$	$(\emptyset, P)$	(OP, EP)

Proof: For the proof and later use, we introduce the canonical element $\omega = \omega_{\lambda\mu}$ indexed by (λ, μ).

Define

$$\omega = \omega_{\lambda\mu} := \omega_1 \omega_2 \cdots \omega'_1 \omega'_2 \cdots$$

where ω_i and ω'_i are

$$\omega_i := \sigma_{a_{i-1}+1} \cdots \sigma_{a_i - 2} \sigma_{a_i - 1} \ (a_i = \lambda_1 + \cdots + \lambda_i),$$

$$\omega'_i := \sigma_{b_{i-1}+1} \cdots \sigma_{b_i - 2} \tau_{b_i} \ (b_i = |\lambda| + \mu_1 + \cdots + \mu_i).$$

Let $\omega \longmapsto \hat{\omega}$ denote the canonical epimorphism $W_n[\alpha] \longrightarrow W_n$. Then $\hat{\omega}_{\lambda\mu}$ belongs to the class indexed by (λ, μ). Note that $\hat{\omega}_i$ (rep., $\hat{\omega}'_i$) is a positive λ_i-cycle (rep., negative μ_i-cycle).

The theorem then follows from the proof of Stembridge's Theorem 2.1 [23].

■

2.4 Clifford Theory for $\mathbf{Z}_2^2$-Quotients

Let G be a finite group with a subgroup H of index 2, and let ν denote the "sign" character of the natural homomorphism $G \longrightarrow G/H$. Let $\mathbf{K}$ be a field of characteristic zero. If V is an $\mathbf{K}G$-module, we will say that V is self-associate (with respect to ν) if $V \simeq \nu \otimes V$; otherwise, we will say that V and $\nu \otimes V$ form an associate pair (with respect to ν).

If V is self-associate, there exists an endomorphism $S \in GL_{\mathbf{K}}(V)$ such that

$$gSv = \nu(g)Sgv$$

for all $v \in V, g \in G$. We will refer to S as an ν-associator of V.

We now consider a similar analysis for the case in which G has a normal subgroup H such that $G/H \simeq \mathbf{Z}_2 \times \mathbf{Z}_2$. The principal examples for G we have in mind are W_n and its double covers $W_n[\alpha]$.

Let $L_n \simeq Hom(G/H, \mathbf{K}^\times)$ denote the group of four linear characters of G that arise from the quotient of G by H, and let α be any factor set of G. Note that L_n acts on (isomorphism classes of) $\mathbf{K}G^\alpha$-modules via $V \longmapsto \nu \otimes V$ $(\nu \in L_n)$. We will use the notation $L_V = \{\nu \in L_n : V \simeq \nu \otimes V\}$ to denote the stabilizer of V.

Before we go any further, we like to spend some time talking about the definition of ν-associator. In the following we simply use S_V or S for a ν-associator if there is no confusion from the context.

From the definition of ν-associator, we immediately have the following results:

1. $S_V \in End_{\mathbf{K}H}(V)$.

2. Let H_ν be the kernel of ν, we have $S_V \in End_{\mathbf{Q}H_\nu}(V)$.

3. Define

$$
\begin{aligned}
A_0 &= \{f \in End_{\mathbf{Q}H}(V) : gf = fg, \forall g \in G\} \\
A_1 &= \{f \in End_{\mathbf{Q}H}(V) : gf = -fg, \forall g \in G \setminus H_\nu\}.
\end{aligned}
$$

Then $S \in A_1$ and any invertible element of A_1 can be chosen to be a ν-associator of V.

4. We need to mention the differences between our definition and Stembridge's definition in [23]. All modules in Stembridge's definition are irreducible and the base field is $\mathbf{C}$. So by Schur's Lemma, S^2 is always a scalar and S can always be modified to ensure $S^2 = 1$.

This is different from our definition. S^2 may not always be a scalar since our base field is $\mathbf{K}$ (particularly $\mathbf{Q}$) not $\mathbf{C}$. This will become more clear when we go to Chapter Four. Next, the modules in our definition may not be irreducible. In our applications, S can be chosen to make sure S^2 is a scalar, but we can not modify S to ensure $S^2 = 1$. Finally, assume S_1, S_2 are two ν-associators; even in the situation when both S_1^2, S_2^2 are scalar, they may be quite different, not like what might be expected that they differ only by a square in $\mathbf{K}^\times$.

The following result classifies the possible behaviors that can occur when a $\mathbf{C}G^\alpha$-module is restricted to $\mathbf{C}H^\alpha$.

Theorem 2.4.1 *Let V be a finite dimensional, irreducible $\mathbf{C}G^\alpha$-module.*

(a) If $L_V = \{1\}$, then V_H is an irreducible $\mathbf{C}H^\alpha$-module.

(b) If $L_V = \{1, \varepsilon\}$, then V_H is the sum of two irreducible, non-isomorphic $\mathbf{C}H^\alpha$-modules (and similarly for $L_V = \{1, \delta\}$ and $L_V = \{1, \varepsilon\delta\}$).

(c) If $L_V = \{1, \varepsilon, \delta, \varepsilon\delta\}$, and $S, T \in GL(V)$ are the associators of V for ε and δ, then $ST = \pm TS$. Moreover,

(i) If $ST = TS$, then V is the direct sum of 4 irreducible, non-isomorphic $\mathbf{C}H^{\alpha}$-modules.

(ii) If $ST = -TS$, then V is a direct sum of two copies of one irreducible $\mathbf{C}H^{\alpha}$-modules.

Proof: Since the base field is $\mathbf{C}$, we can always find complex numbers a, b such that $(aS)^2 = 1$, $(bT)^2 = 1$. Let $S' = aS, T' = bT$, then S' and T' are the ε-associator and the δ-associator, respectively. In other words, V has an ε-associator and a δ-associator in terms of Stembridge's definition in [23]. Then results follow from his Theorem 3.1. ∎

Any group G with a $\mathbf{Z}_2^2$-quotient has a two-dimensional projective representation arising from the fact that the dihedral group of order 8 doubly covers $\mathbf{Z}_2^2$. To be more precise, let $\Theta_0 : \mathbf{Z}_2^2 \longrightarrow PGL_2(\mathbf{Q})$ denote the projective representation obtained from the reflection representation of the dihedral group modulo its center. We may thus obtain a projective G-representation, also to be denoted by Θ_0, via

$$G \longrightarrow G/H \longrightarrow \mathbf{Z}_2^2 \longrightarrow GL_2(\mathbf{Q}).$$

In our applications, the representation Θ_0 can be chosen to be self-associate with respect to all linear characters $1, \varepsilon, \delta, \varepsilon\delta$. Let β denote the factor set of Θ_0, observe that if V is any $\mathbf{C}G^{\alpha}$-module, then the twisted tensor product $\Theta_0 \otimes V$ is a $\mathbf{C}G^{\beta\alpha}$-module. Here $\Theta_0 \otimes V$ means $V_0 \otimes V$ where V_0 is a $\mathbf{C}G^{\beta}$-module affording Θ_0. The following result classifies the irreducible constituents of $\Theta_0 \otimes V$.

Theorem 2.4.2 *Let V be a finite-dimensional, irreducible $\mathbf{C}G^{\alpha}$-module.*

(a) If $L_V = \{1\}$, then $\Theta_0 \otimes V$ is an irreducible $\mathbf{C}G^{\beta\alpha}$-module.

(b) If $L_V = \{1, \varepsilon\}$, then $\Theta_0 \otimes V$ is a direct sum of two irreducible, non-isomorphic $\mathbf{C}G^{\beta\alpha}$-modules (and similarly for $L_V = \{1, \delta\}$ and $L_V = \{1, \varepsilon\delta\}$).

(c) If $L_V = \{1, \varepsilon, \delta, \varepsilon\delta\}$, then let S denote the ε-associator, and T denote the δ-associator for V.

(i) If $ST = TS$, the $\Theta_0 \otimes V$ is a direct sum two copies of one irreducible $\mathbf{C}G^{\beta\alpha}$*-module*

(ii) If $ST = -TS$, the $\Theta_0 \otimes V$ is a direct sum of 4 irreducible, non-isomorphic $\mathbf{C}G^{\beta\alpha}$*-modules.*

Proof: Since the base field is $\mathbf{C}$, we can always find complex numbers a, b such that $(aS)^2 = 1$, $(bT)^2 = 1$. Let $S' = aS, T' = bT$, then S' and T' are an ε-associator and a δ-associator, respectively. In other words, V has ε-associator and δ-associator in terms of Stembridge's definition in [23]. Then results follow from his Theorem 3.2. ■

Remark : In this section we defined the "associator" of a module, defined a special spin representation Θ_0, and discussed the irreducible constituents of the twisted product $\Theta_0 \otimes M$ where M is a $\mathbf{C}W_n[\beta]$-module. We have more to say about these definitions and the above theorems. We will come back to this subject in Section 3.3.

In general, let Θ be a representation of $W_n[\beta]$ and also assume that Θ is self-dual. Let V be any $\mathbf{C}W_n[\alpha]$-module, then $\Theta \otimes V$ is a $\mathbf{C}W_n[\beta\alpha]$-module. Therefore, the twisted tensor product $\Theta \otimes V$ permits us to easily create a large supply of representations of the double cover $W_n[\beta\alpha]$. The following result shows that all irreducible representation of the group $W_n[\beta\alpha]$ are of this form.

Theorem 2.4.3 *Every irreducible spin $\mathbf{C}W_n[\beta\alpha]$-module V is (isomorphic to) a submodule of $\Theta \otimes V'$, for some spin $\mathbf{C}W_n[\alpha]$-module V'.*

Proof: Let χ be an irreducible character of $W_n[\beta\alpha]$ afforded by V and let θ be the character of $W_n[\beta]$ afforded by Θ. Then $V' = \Theta \otimes V$ is a $\mathbf{C}W_n[\alpha]$-module affording the twisted product $\theta\chi$ and $\Theta \otimes V' = \Theta \otimes \Theta \otimes V$ affords the twisted product $\theta(\theta\chi)$.

Note that V' is a $\mathbf{C}W_n[\alpha]$-module and that V is a submodule of $\Theta \otimes V'$. It is equivalent to show $\langle \chi, \theta\theta\chi \rangle \neq 0$. In fact, note that θ is self-dual, $\bar{\theta} = \theta$. It then follows that

$$\begin{aligned} \langle \chi, \theta\theta\chi \rangle &= < \bar{\theta}\chi, \theta\chi > \\ &= < \theta\chi, \theta\chi > \\ &\neq 0. \end{aligned}$$

This completes the proof of the Theorem. ∎

By Theorem 2.4.3, a complete list of irreducible $\mathbf{C}W_n[\beta\alpha]$-modules can be constructed by choosing one irreducible $\mathbf{C}W_n[\alpha]$-module V' from each L_n-orbit, and decomposing $\Theta \otimes V'$ into irreducible $\mathbf{C}W_n[\beta\alpha]$-modules. In Chapter Four, we will apply this technique to the spin representations of the double covers of W_n.

2.5 Quaternion Algebras and Brauer-Wall Groups

In order to calculate the Schur indices we will need a limited amount of computation in the Brauer group. We begin this section by describing how to effectively perform these computations. Let $\mathbf{K}$ be some field of characteristic zero. We denote by $Br(\mathbf{K})$ the Brauer group of $\mathbf{K}$. If $a, b \in \mathbf{K}^\times$, we denote by (a, b) the class in $Br(\mathbf{K})$ of the quaternion $\mathbf{K}$-algebra with generators i and j satisfying $i^2 = a,\ j^2 = b$ and $ij = -ji$. We record some basic properties which can be found in Chapter 3 of Lam [12].

Proposition 2.5.1 *Some basic properties of quaternion algebras:*

(1) $(a, b) = (ax^2, by^2) = (b, a)$, *if* $a, b, x, y \in \mathbf{K}^\times$.

(2) For $a, b \in \mathbf{K}^\times,\ (a, b) = 1$ *iff there exist* $x, y \in \mathbf{K}$ *with* $ax^2 + by^2 = 1$.

(3) For $a, b, c \in \mathbf{K}^\times,\ (a, b)(a, c) = (a, bc)$.

(4) For $a \in \mathbf{K}^\times,\ (a, 1) = (a, -a) = 1$.

Proof: Refer to the appropriate result in Chapter 3, Section 18.2 of Lam [12].

■

Our computations in some cases will need some elementary results on graded algebras and Brauer-Wall groups. So in the following, we will introduce our notation for them and some results which we shall need. Our notation here follows mostly that of Lam [12]. Thus, by a graded algebra A over $\mathbf{K}$ we mean a $\mathbf{Z}_2$-graded algebra over $\mathbf{K}$. As a $\mathbf{K}$-vector space, $A = A_0 \oplus A_1$, the direct sum of the elements of degree 0 with the elements of degree 1. The elements of $h(A) = A_0 \cup A_1$ are called the homogeneous elements of A. If $a \in h(A)$, we write $\partial(a) = i$ if $a \in A_i$ $(i = 0, 1)$. This degree function ∂ is not well-defined at 0, but, in practice, this does not create undue difficulty. The center $Z(A)$ is a graded sub-algebra of A. The graded center $\hat{Z}(A)$ is the graded sub-algebra spanned by all $c \in h(A)$ such that $cs = (-1)^{\partial c \partial s} sc$ for all $s \in h(A)$. Notice that if $s \in A_1$ and $s^2 \neq 0$ then $s \notin \hat{Z}(A)$. A central graded algebra over $\mathbf{K}$ is one where $\hat{Z}(A)$ is one dimensional over $\mathbf{K}$ (We write $\hat{Z}(A) = \mathbf{K}$). A simple graded algebra is one without any proper two-sided graded ideals. If in a addition it is a central graded algebra over $\mathbf{K}$, we say it is a central simple graded algebra (**CSGA**) over $\mathbf{K}$. The tensor product $A \otimes B$ of graded algebras A and B over $\mathbf{K}$ is another graded algebra over $\mathbf{K}$. We call the same set $A \otimes B$ with regular addition and multiplication induced by

$$(a \otimes b)(a' \otimes b') = (-1)^{\partial b \partial a'} aa' \otimes bb' \ \ (a, a' \in h(A), b, b' \in h(B))\,.$$

the graded tensor product of A and B, and we denote it by $A \hat{\otimes} B$. The point is that the graded tensor product of central simple graded algebras is necessarily a central simple graded algebra. It is easy to check that, up to isomorphism, $\hat{\otimes}$ is associative and commutative. $a \hat{\otimes} b \longmapsto (-1)^{\partial a \partial b} b \hat{\otimes} a$ provides an isomorphism $A \hat{\otimes} B \longrightarrow B \hat{\otimes} A$. A central simple graded algebra A over $\mathbf{K}$ is said to be of even type if A is a central simple algebra over $\mathbf{K}$ (as an ungraded algebra). Otherwise, we say it is of odd type.

Theorem 2.5.2 *Let* $\mathbf{K}$ *be a field of characteristic zero and* A *be a central simple graded algebra over* $\mathbf{K}$ *of odd type. Then* $Z(A) = \mathbf{K} \oplus \mathbf{K}z$*, where* $z \in A_1$ *and* $z^2 = a \in \mathbf{K}^\times$*. The square class of* a *does not depend on the choice of* $z \in Z_1^\times$*. Furthermore,* A_0 *is a central simple algebra over* $\mathbf{K}$ *and* A *is a direct sum of one or two central simple algebras over* $\mathbf{K}(\sqrt{a})$*, each of which, as an element of* $Br\,(\mathbf{K}(\sqrt{a}))$*, yields simply the extension of scalars of* A_0*, namely* $A_0 \otimes \mathbf{K}(\sqrt{a})$*.*

Proof: The result follows from Proposition 3.3 , p.86, and Theorem 3.6, p.88, in Lam [12]. ■

Theorem 2.5.3 *Let* $\mathbf{K}$ *be a field of characteristic zero and* A *be a central simple graded algebra over* $\mathbf{K}$ *of even type. Then there exists* $z \in Z(A_0)$ *such that* $Z(A_0) = \mathbf{K} \oplus \mathbf{K}z$ *and* $z^2 = a \in \mathbf{K}^\times$*. The element* z *is determined up to a scalar multiple by theses properties, hence the square class of* a *is uniquely determined. Furthermore,* A_0 *is the direct sum of one or two central simple algebras over* $\mathbf{K}(\sqrt{a})$*, each of which, as an element of* $Br\,(\mathbf{K}(\sqrt{a}))$*, is simply the extension of scalars of* A*, namely* $A \otimes \mathbf{K}(\sqrt{a})$*.*

Proof: If $A_1 = 0$ then set $z = 1$ and a is a square in $\mathbf{K} \setminus \{0\}$. Hence $\mathbf{K}(\sqrt{a}) = \mathbf{K}$ and the result holds. If $A_1 \neq 0$, the result follows from Theorem 3.8, p.89, in Lam [12]. ■

To every central simple graded algebra A over $\mathbf{K}$, we associate now a triple $[A] = [D, t, a] \in Br(\mathbf{K}) \times \mathbf{Z}_2 \times (\mathbf{K}^\times)/(\mathbf{K}^\times)^2$. The invariant t is called the type of A and $t = 0$ if A is of even type and $t = 1$ if A is of odd type. If A is of even type, Theorem 2.5.3 defines $a \in \mathbf{K}^\times$ up to a square and, similarly, Theorem 2.5.2 defines $a \in \mathbf{K}^\times$ up to a square if A is of odd type. This gives the third coordinate in the triple which is called the quadratic invariant of A. If A is of even type, then A is a central simple algebra over $\mathbf{K}$ (when the grading is ignored), so we set D, the first coordinate in the triple, to be the class of A in $Br(\mathbf{K})$ in this case. If A is of odd

type, then by Theorem 2.5.2, A_0 is a central simple algebra over $\mathbf{K}$ and we set D, the first coordinate in the triple, to be the class of A_0 in $Br(\mathbf{K})$ in this case.

Theorem 2.5.4 *The map described above that associates a triple in* $Br(\mathbf{K}) \times \mathbf{Z}_2 \times (\mathbf{K}^\times)/(\mathbf{K}^\times)^2$ *to every central simple graded algebra over* $\mathbf{K}$*, yields a one-to-one correspondence between* $Br(\mathbf{K}) \times \mathbf{Z}_2 \times (\mathbf{K}^\times)/(\mathbf{K}^\times)^2$ *and* $BW(\mathbf{K})$*. Hence the group structure on* $BW(\mathbf{K})$ *gives a group structure on the set of triples denoted by juxtaposition. Furthermore, if* $D, E \in Br(\mathbf{K})$ *and* $a, b \in (\mathbf{K}^\times)/(\mathbf{K}^\times)^2$*, then the following hold:*

(1) $[D, 1, a][E, 1, b] = [DE(a, b), 0, -ab]$,

(2) $[D, 0, a][E, 0, b] = [DE(a, b), 0, ab]$,

(3) $[D, 0, a][E, 1, b] = [DE(a, -b), 1, ab]$,

(4) $[D, 0, a]^{-1} = [D^{-1}(a, a), 0, a]$,

(5) $[D, 1, a]^{-1} = [D^{-1}, 1, -a]$.

Proof: See Theorem 3.9 and Theorem 3.10 on page 119 of Lam [12]. ■

Lemma 2.5.5 *Let* $\mathbf{K}$ *be a field of characteristic zero, and* A *be a central simple graded algebra over* $\mathbf{K}$ *of even type. Assume, as an ungraded algebra,* $A \simeq M_n(D)$*, where* D *is a quaternion algebra. Let, from Theorem 3.8 of [12]*

$$\begin{aligned} A &\simeq [D, 0, a], \ A = A_0 + A_1, \\ Z(A_0) &\simeq \mathbf{K} + i\mathbf{K}, \ i^2 = a \in \mathbf{K}^\times. \end{aligned}$$

Furthermore, assume $dim_{\mathbf{K}} A = 2 dim_{\mathbf{K}} A_0$*. Then there exists some* $j \in A_1$ *such that*

$$ij = -ji, \ j^2 \in \mathbf{K}^\times$$

and

$$A \simeq C_A(i, j) \otimes (i^2, j^2), \ C_A(i, j) \subseteq A_0,$$

where (i^2, j^2) *is a quaternion algebra generated by* i *and* j*, and* $C_A(i, j)$ *denotes the centralizer of* i *and* j*, which is a central simple algebra.*

Proof: Our proof is based on the proof of the Theorem 3.8 of Lam [12]. We first show that there exists some $j \in A_1$ such that $ij = -ji$, $j^2 \in \mathbf{K}^\times$, so A contains a quaternion algebra (i^2, j^2) generated by i and j. We need to consider three cases.

Suppose first $a \in (\mathbf{K}^\times)^2$. By the proof of Theorem 3.8 of Lam [12], we may assume

$$i = \begin{pmatrix} -I_r & 0 \\ 0 & I_s \end{pmatrix},$$

and identify

$$A_0 = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}$$

and

$$A_1 = \begin{pmatrix} 0 & P \\ Q & 0 \end{pmatrix}.$$

Furthermore, the assumption $dim_{\mathbf{K}}A = 2dim_{\mathbf{K}}A_0$ implies

$$n = r + s, \text{ and } n^2 = 2(r^2 + s^2).$$

It is easy to see from the above two identities that

$$r = s = \frac{n}{2}.$$

Then j can be chosen to be

$$\begin{pmatrix} 0 & I_r \\ I_r & 0 \end{pmatrix},$$

so j exists in this case.

Suppose next that $a \notin (\mathbf{K}^\times)^2$, and the field $Z(A_0) = \mathbf{K}(\sqrt{a})$ can be embedded into D. By the proof of Theorem 3.8 of Lam [12], we may put a grading on D in such a way that $i \in D_0$, $A_0 = M_n(D_0)$, and $A_1 = M_n(D_1)$. In this way, since D is a quaternion algebra and is embedded in A, the existence of j is obvious.

Finally suppose $a \notin (\mathbf{K}^{\times})^2$, and the field $Z(A_0) = \mathbf{K}(\sqrt{a})$ can not be embedded into D. By the proof of Theorem 3.8 of Lam [12], we may assume that $A \simeq M_{2m}(D)$ and identify i to be a matrix with diagonal blocks of the form

$$e = \begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix}.$$

Also we may assume that A_0 consists of matrices in block form $(M_{i,j}), 1 \leq i, j \leq m$, where $M_{i,j}$ are 2×2 matrices, each commuting with e. Similarly, A_1 consists of $(N_{i,j})$, where $N_{i,j} \cdot e = -e \cdot N_{i,j}$. Then it is easy to see that if we take j to be the matrix with only diagonal blocks of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

then $j \in A_1, j^2 \in \mathbf{K}^{\times}$. So j exists in this case also. Thus, the proof of the existence of j is finished.

Next by the proof of Theorem 3.8 in Lam [12], we have $A_0 = C_A(i)$. This forces $C_A(i,j) \subseteq A_0$. Since $< i, j >$ is a quaternion algebra, by (iv) of Theorem 2.8.3, $C_A(i,j)$ is a central simple algebra and $A \simeq C_A(i,j) \otimes < i, j >$.

Thus, the proof of the lemma is now finished. ■

Theorem 2.5.6 *Let* $\mathbf{K}$ *be a field of characteristic zero,* A *be a central simple graded algebra over* $\mathbf{K}$ *of even type with* $dim_{\mathbf{K}} A = 2 dim_{\mathbf{K}} A_0$*, and assume* $A \simeq M_n(D)$ *where* D *is a quaternion algebra and*

$$\begin{aligned} A &\simeq [D, 0, a], \ A = A_0 + A_1, \\ Z(A_0) &= \mathbf{K} + i\mathbf{K}, \ i^2 = a \in \mathbf{K}^{\times}. \end{aligned}$$

Further assume B *is a* $\mathbf{K}$*-algebra and* b *is an element of* B *such that* $b^2 \in \mathbf{K}^{\times}$*. Consider the sub-algebra* P *of* $A \otimes B$*, where* P *is generated by* $A_0 \otimes 1$ *and* $j \otimes b$*, i.e.,*

$$P = \langle A_0 \otimes 1, j \otimes b \rangle,$$

here j is the element we found in Lemma 2.5.5 above.

Then P is a central simple $\mathbf{K}$-algebra and

$$P \simeq C_A(i,j) \otimes (i^2, j^2b^2).$$

Furthermore

$$[P] = D(a, b^2)$$

in $Br(\mathbf{K})$.

Proof: By Lemma 2.5.5, A has a quaternion algebra (i^2, j^2) for some $j \in A_1$, and a central simple algebra $C_A(i,j) \subseteq A_0$ such that $A \simeq C_A(i,j) \otimes (i^2, j^2)$ and $[A] = D(i^2, j^2)$ in $Br(\mathbf{K})$. This implies that $< i \otimes 1, j \otimes b >$, a quaternion algebra, is a sub-algebra of P. Furthermore, $P' := C_A(i,j) \otimes 1$, which is isomorphic to $C_A(i,j)$, is a central simple subalgebra of P, and P' is contained in the centralizer of $< i \otimes 1, j \otimes b >$. By a dimension count we get

$$\begin{aligned} P &\simeq P' \otimes < i \otimes 1, j \otimes b > \\ &\simeq C_A(i,j) \otimes (i^2, j^2b^2) \end{aligned}$$

and consequently

$$\begin{aligned} [P] &= [C_A(i,j)](i^2, j^2)(i^2, b^2) \\ &= [A](i^2, b^2) \\ &= D(a, b^2) \end{aligned}$$

in $Br(\mathbf{K})$.

The proof of theorem is now complete. ■

Corollary 2.5.7 *Under the assumptions of the above Theorem, if one further assumes $b^2 = 1$, then*

$$< A_0 \otimes 1, j \otimes b > \simeq A.$$

Proof: By the above theorem we have

$$\begin{aligned} < A_0 \otimes 1, j \otimes b > &\simeq C_A(i,j) \otimes (i^2, j^2b^2) \\ &\simeq C_A(i,j) \otimes (i^2, j^2). \end{aligned}$$

It then follows from Lemma 2.5.5

$$\begin{aligned} < A_0 \otimes 1, j \otimes b > &\simeq C_A(i,j) \otimes (i^2, j^2) \\ &\simeq A. \end{aligned}$$

The result is proved. ■

2.6 Schur Indices and Group Pairs

Let $\mathbf{K}$ be a field of characteristic zero. Let χ be an irreducible character of some finite group G. Then we denote by $\mathbf{K}(\chi)$ the extension of $\mathbf{K}$ by the values $\{\chi(g) : g \in G\}$. Let $\chi_1 = \chi,\ \chi_2, \ldots,\ \chi_r$ be the distinct conjugates of χ under the Galois group $Gal\,(\mathbf{K}(\chi)/\mathbf{K})$. The Schur index of χ with respect to $\mathbf{K}$ is the smallest positive integer $m_{\mathbf{K}}(\chi)$ such that $m_{\mathbf{K}}(\chi)(\chi_1 + \cdots + \chi_r)$ is a character afforded by some $\mathbf{K}G$-module M. We record a well-known relationship between the Schur index and the Brauer group as follows.

Proposition 2.6.1 *With the above notation, $m_{\mathbf{K}}(\chi) = m_{\mathbf{K}(\chi)}(\chi)$. Furthermore, let N be any $\mathbf{K}(\chi)$-module with character $n\chi$, for some positive integer n. Then $End_{\mathbf{K}(\chi)G}(N)$ is a central simple algebra over $\mathbf{K}(\chi)$ of dimension n^2 which determines an element of $Br(\mathbf{K}(\chi))$ not dependent on N (which we henceforth denote by $[\chi]$). If Δ is the division algebra which also represents $[\chi]$ then $dim_{\mathbf{K}(\chi)}(\Delta) = m_{\mathbf{K}}(\chi)^2$; that is, $m_{\mathbf{K}}(\chi)$ is the rank of Δ.*

Proof: See, for example, Lemma 2.1 of Turull [24]. ■

Remark : We will call $[\chi]$ the class of central simple algebras associated with χ, as an element of $Br(\mathbf{K}(\chi))$.

Theorem 2.6.2 *Let* $\mathbf{K}$ *be a field of characteristic zero. Let* χ *be an irreducible character of some finite group* G*. Let* $\chi_1 = \chi, \cdots, \chi_r$ *be the distinct conjugates of* χ *under* $Gal(\mathbf{K}(\chi)/\mathbf{K})$*. Suppose* $m(\chi_1 + \cdots + \chi_r)$ *can be afforded by a* $\mathbf{K}G$*-module* M*, where* m *is a rational integer, then*

$$A_G(M) := End_{\mathbf{K}G}(M)$$

is a simple $\mathbf{K}$*-algebra with* $Z(A_G(M)) \simeq \mathbf{K}(\chi)$*.*

Furthermore, $A_G(M)$ *can be viewed as a central simple algebra over* $\mathbf{K}(\chi)$ *and* $[\chi] = A_G(M)$ *in* $Br(\mathbf{K}(\chi))$*.*

Proof: This follows immediately from above proposition. ■

Remark : We will call $A_G(M) := End_{\mathbf{K}G}(M)$ the simple algebra associated with character χ.

In the rest of the dissertation, we will frequently use the notation $A_0(M)$ for $A_G(M)$ where M is a $\mathbf{Q}G$-module.

It is the purpose of this dissertation to explicitly calculate $[\chi]$, for all spin characters χ of $W_n[\alpha]$. This turns out to be $A_0(M)$, viewed as a central simple algebra over $\mathbf{K}(\chi)$, as an element of Brauer group $Br(\mathbf{K}(\chi))$. Furthermore we can calculate the rank of the division algebra corresponding to $[\chi]$. (This rank turns out to be always 1 or 2.)

Let H_1 be the kernel of ε, then H_1 is a subgroup of $W_n[\alpha]$ of index two. Similarly, if H_2 is the kernel of δ, then H_2 is another subgroup of $W_n[\alpha]$ of index two. So, it is convenient for us to introduce the following definition.

Definition 2.6.3 *A group pair is a pair* (G, H) *where* G *is a finite group and* H *is a subgroup of* G *of index two. We may write* G *for the group pair* (G, H) *if* H *is understood from the context.*

Let ε denote the "sign" character of G with respect to the subgroup H, and let χ be any irreducible (complex) character of G. Then we have the following simple result.

Lemma 2.6.4 *With the above notation, $\chi|_H$ is irreducible if and only if χ is not self-associate, i.e., $\varepsilon\chi \neq \chi$.*

Proof: If $\chi|_H = \theta \in Irr(H)$, by Gallagher's Theorem (6.17 of Isaacs [9]), $\varepsilon\chi \neq \chi$.

Conversely, assume $\varepsilon\chi \neq \chi$, we want to show $\chi|_H$ is irreducible. Suppose not, by Clifford's theorem, we have

$$\chi|_H = \theta + \theta', \theta \neq \theta' \in Irr(H)$$

such that θ and θ' are G-conjugate. It follows that $H = I_G(\theta)$. Therefore, by (6.11) of Isaacs [9], χ is the only irreducible character of G with $[\chi|_H, \theta] \neq 0$. Since $(\varepsilon\chi)|_H = \chi|_H$, we must have $\chi = \varepsilon\chi$, a contradiction.

The proof is now complete. ■

Definition 2.6.5 *Let (G, H) be a group pair and let M be any $\mathbf{K}G$-module. We denote by $A(M)$ the following graded $\mathbf{K}$-algebra: As an ungraded algebra $A(M) = End_{\mathbf{K}H}(M)$. Furthermore $A(M)_0 = End_{\mathbf{K}G}(M)$ and $A(M)_1 = \{f \in A(M) :$ if $x \in G \setminus H$ then $xfx^{-1} = -f\}$.*

Lemma 2.6.6 *Let (G, H) be a group pair and let M be an irreducible $\mathbf{K}G$-module. Let $m(\chi_1 + \ldots + \chi_r)$ (where $\chi = \chi_1, \ldots, \chi_r$ are irreducible characters of G) be the character afforded by M. Then $A(M)$ is a simple graded K-algebra. Furthermore, $A(M)$ is a central simple graded $\mathbf{K}$-algebra if and only if $\mathbf{K}(\chi|_H) = \mathbf{K}$ (that is, $\chi(h) \in \mathbf{K}$ for all $h \in H$).*

Proof: This is just Lemma 3.4 of Turull [25]. ■

Definition 2.6.7 *Let (G, H) be a group pair and let χ be some irreducible character of G. We say that a $\mathbf{K}G$-module M is χ-quasihomogeneous if the character φ afforded by M is such that $\varphi|_H$ is a rational multiple of $\chi|_H$.*

Theorem 2.6.8 *Let (G, H) be a group pair and let $M \neq 0$ be a $\mathbf{K}G$-module. Then $A(M)$ is a central simple graded K-algebra if and only if M is χ-quasihomogeneous for some irreducible character χ of G.*

Proof: This is just Theorem 3.6 of Turull [25]. ■

Corollary 2.6.9 *Let (G, H) be a group pair and let χ be an irreducible character of G. Suppose that $\mathbf{K}(\chi|_H) = \mathbf{K}$. Then the element in $BW(\mathbf{K})$ corresponding to the central simple graded algebra $A(M)$, where $M \neq 0$ is a χ-quasihomogeneous $\mathbf{K}G$-module, is independent of M. This element of $BW(\mathbf{K})$ is denoted by $[[\chi]]$.*

Proof: This is just Corollary 3.7 of Turull [25]. ■

The invariant $[[\chi]]$ describes the invariant $[\chi]$ of Proposition 2.6.1, and hence the Schur index $m_{\mathbf{K}}(\chi)$, for the irreducible character associated with χ. We detail this in the next proposition.

Proposition 2.6.10 *Let (G, H) be a group pair and let χ be an irreducible character of G with $\mathbf{K}(\chi|_H) = \mathbf{K}$. Let $[D, t, a]$ be the triple associated with $[[\chi]]$ in Corollary 2.6.9. Then exactly one of the following holds:*

(1) $\varepsilon\chi \neq \chi, t = 0$ and $[\chi|_H] = D$. Denote $\varepsilon\chi$ by χ'. Then $\mathbf{K}(\chi) = \mathbf{K}(\chi') = \mathbf{K}(\sqrt{a})$. Furthermore, $[\chi] = [\chi'] = D \otimes \mathbf{K}(\sqrt{a})$ in $Br\,(\mathbf{K}(\sqrt{a}))$.

(2) $\varepsilon\chi = \chi, t = 1, \mathbf{K}(\chi) = \mathbf{K}$ and $[\chi] = D$ in $Br(\mathbf{K})$. Let $\chi|_H = \theta_1 + \theta_2$. Then $\theta_1 \neq \theta_2$ are irreducible characters of H and $\mathbf{K}(\theta_1) = \mathbf{K}(\theta_2) = \mathbf{K}(\sqrt{a})$ and $[\theta_1] = [\theta_2] = D \otimes \mathbf{K}(\sqrt{a})$ in $Br\,(\mathbf{K}(\sqrt{a}))$.

Proof: This is just Proposition 3.8 of Turull [25]. ■

2.7 Group Pairs with a Distinguished Central Involution

The groups we are interested in, namely $G = W_n[\alpha]$, not only have a distinguished subgroup $H = ker(\varepsilon)$ of index two, making (G, H) a group pair, but also they have a distinguished involution $-1 \in H$. Turull [25] considered this case, gave a general definition, developed some fundamental properties along this line, and used this technique effectively to compute the Schur indices for all spin characters of double covers of the symmetric groups S_n. We are going to use the same technique all the time to study the spin representations of double covers of W_n in this dissertation. To make our dissertation to be self-contained and much easier to read, we carry over the whole Section 4 of Turull [25] to here.

Let (G_1, H_1) and (G_2, H_2) be group pairs with a distinguished central involution. We define $G_1 \vee G_2$ to be the group with the following properties: G_1 and G_2 are (isomorphic to) normal subgroups of $G_1 \vee G_2$ and $G_1G_2 = G_1 \vee G_2$ and $G_1 \cap G_2 = \{1, -1\}$. Furthermore, if $g_1 \in G_1$ and $g_2 \in G_2$ then $g_1g_2 = g_2g_1$ unless both $g_1 \notin H_1$ and $g_2 \notin H_2$, in which case $g_1g_2 = (-1)g_2g_1$. The subset $\{g_1g_2 \in G_1 \vee G_2 :$ either both $g_1 \in H_1$ and $g_2 \in H_2$ or both $g_1 \notin G_1 \backslash H_1$ and $g_2 \notin G_2 \backslash H_2\}$ of $G_1 \vee G_2$ is a distinguished subgroup of index two of $G_1 \vee G_2$, and -1 is a distinguished central involution, so that we view $G_1 \vee G_2$ as a group pair with a distinguished central involution. The existence of $G_1 \vee G_2$ is easily verified, but an explicit construction of $G_1 \vee G_2$ is needed and given in the next proposition. It is clear that $G_1 \vee G_2$ is uniquely determined (up to isomorphism) by the group pairs with involution G_1 and G_2, and that the operation $\vee$ is commutative and associative (up to isomorphism). Furthermore, if $\alpha = [1, 1, -1, 1, 1]$ and λ, μ are partitions such that $|\lambda| + |\mu| = n$, say $|\lambda| = k, |\mu| = n - k$, then $W_{k,n-k}[\alpha] = W_k[\alpha] \vee W_{n-k}[\alpha]$ (we will explain and prove this below, see Theorem 3.5.10).

Proposition 2.7.1 *Let (G_1, H_1) and (G_2, H_2) be group pairs with distinguished central involution -1. For $i = 1, 2$, let M_i be a $\mathbf{K}G_i$-module with character χ_i such that*

$(-1)\cdot v = -v$ for all $v \in M_i$. Then there exists a $\mathbf{K}(G_1 \vee G_2)$-module $M_1 \vee M_2$ with character $\chi_1 \vee \chi_2$ where, for $g_1 \in G_1$ and $g_2 \in G_2$,

$$(\chi_1 \vee \chi_2)(g_1 g_2) = \begin{cases} 0 & \text{if either } g_1 \notin H_1 \text{ or } g_2 \notin H_2, \\ 2\chi_1(g_1)\chi_2(g_2) & \text{if both } g_1 \in H_1 \text{ and } g_2 \in H_2. \end{cases}$$

Proof: We begin by giving a convenient construction of $G_1 \vee G_2$. Let D be the dihedral group of order 8, generated by τ_1 and τ_2 where $\tau_1^2 = 1, \tau_2^2 = 1, (\tau_1\tau_2)^2 = -1$ and $(-1)^2 = 1$. The group G_1/H_1 is isomorphic to $< \tau_1 > D'/D'$ and the group G_2/H_2 is isomorphic to $< \tau_2 > D'/D'$; so we have a group homomorphism $\varphi : G_1 \times G_2 \to D/D'$ such that $\ker(\varphi) = H_1 \times H_2$ and $\varphi(g_1, 1) = \tau_1 D'$ if $g_1 \in G_1 \setminus H_1$ and $\varphi(1, g_2) = \tau_2 D'$ if $g_2 \in G_2 \setminus H_2$. We use the bar convention for the projection map $D \to D/D'$. We define the following subgroup of $G_1 \times G_2 \times D$:

$$\Gamma = \{(g_1, g_2, x) \in G_1 \times G_2 \times D : \varphi(g_1, g_2) = \bar{x}\}.$$

Now Γ is a normal subgroup of index 4 of $G_1 \times G_2 \times D$. Let Ξ be the subgroup of Γ generated by $(-1, -1, 1)$ and $(1, -1, -1)$. Then Ξ is a normal subgroup of Γ of order 4. The monomorphism $G_1 \to \Gamma$, defined by $g \to (g, 1, \tau_1)$ if $g \in G_1 \setminus H_1$ and $g \to (g, 1, 1)$ if $g \in H_1$, projects to a monomorphism $G_1 \to \Gamma/\Xi$. Similarly, from $g \to (1, g, \tau_2)$ for $g \in G_2 \setminus H_2$, we obtain a monomorphism $G_2 \to \Gamma/\Xi$. It is straightforward to check that this provides an isomorphism $\Gamma/\Xi \simeq G_1 \vee G_2$. The distinguished involution is the coset of any one of $(-1, 1, 1), (1, -1, 1)$ or $(1, 1, -1)$. The distinguished subgroup of index two of Γ/Ξ is Ω/Ξ where

$$\Omega = \left\{ \begin{array}{lll} (g_1, g_2, x) \in \Gamma : & \text{either} & \text{both } g_1 \in H_1 \text{ and } g_2 \in H_2 \\ & \text{or} & \text{both } g_1 \notin H_1 \text{ and } g_2 \notin H_2 \end{array} \right\}.$$

Let N be an irreducible faithful $\mathbf{K}D$-module. Then its character ψ is such that $\psi(1) = 2$, $\psi(-1) = -2$ and $\psi(x) = 0$ if $x \in D \setminus D'$. Since the subgroup of $GL(2, \mathbf{K})$ generated by

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is isomorphic to D, the module N exists over any field $\mathbf{K}$ of characteristic zero. The tensor product

$$M_1 \otimes M_2 \otimes N$$

is a $\mathbf{K}(G_1 \times G_2 \times D)$-module, which we restrict to a $\mathbf{K}\Gamma$-module. Since for $-1 \in G_i, (-1) \cdot v = -v$ for all $v \in M_i$ and for $-1 \in D, (-1) \cdot v = -v$ for all $v \in N$, Ξ acts trivially on $M_1 \otimes M_2 \otimes N$. So we view $M_1 \otimes M_2 \otimes N$ as a $\mathbf{K}[G_1 \vee G_2]$-module. We denote this module by $M_1 \vee M_2$. The character $\chi_1 \otimes \chi_2 \otimes \psi$ of the $\mathbf{K}(G_1 \times G_2 \times D)$-module $M_1 \otimes M_2 \otimes N$ is given by

$$(\chi_1 \otimes \chi_2 \otimes \psi)(g_1, g_2, x) = \chi_1(g_1)\chi_2(g_2)\psi(x)$$

for all $g_1 \in G_1, g_2 \in G_2$ and $x \in D$.

Since $\psi(x) = 0$ whenever $x \in D \setminus D'$, it follows that if $(g_1, g_2, x) \in \Gamma$, $(\chi_1 \otimes \chi_2 \otimes \psi)(g_1, g_2, x) = 0$ unless $(g_1, g_2) \in H_1 \times H_2$. Furthermore, if $(g_1, g_2) \in H_1 \times H_2$ then $(g_1, g_2, 1) \in \Gamma$ and $(\chi_1 \otimes \chi_2 \otimes \psi)(g_1, g_2, 1) = 2\chi_1(g_1)\chi_2(g_2)$. We set $\chi_1 \vee \chi_2$ to be the character of $M_1 \vee M_2$. Then the monomorphisms $G_1 \to \Gamma/\Xi$ and $G_2 \to \Gamma/\Xi$ and the previous computation yield the stated values of $\chi_1 \vee \chi_2$ on $G_1 \vee G_2$. This concludes the proof of the proposition. ∎

Remark. From the construction of characters $\chi_1 \vee \chi_2$, we see that the operation $\vee$, on modules or on characters, is commutative and associative (up to appropriate isomorphisms). Hence we can define $(G_1 \vee \cdots \vee G_r)$-modules of the form $M_1 \vee \cdots \vee M_r$. These are closely related to the Clifford products mentioned by Stembridge, p.104 in [22].

Theorem 2.7.2 *Let (G_1, H_1) and (G_2, H_2) be group pairs with distinguished central involution -1. For $i = 1, 2$, let M_i be a $\mathbf{K}G_i$-module with character χ_i such that $(-1) \cdot v = -v$ for all $v \in M_i$. Then, as graded algebras*

$$A(M_1 \vee M_2) \simeq A(M_1)\hat{\otimes}A(M_2)\hat{\otimes}\mathbf{K}[i],$$

where $\mathbf{K}[i]$ is the two-dimensional graded algebra with $(\mathbf{K}[i])_1 = \mathbf{K}i$ and $i^2 = -1$.

Proof: This is just Proposition 4.1 of Turull [25]. ∎

Corollary 2.7.3 *Let* $(G_i, H_i), i = 1, ..., r$ *(with* $r \geq 2$*), be group pairs with distinguished central involution* -1. *For* $i = 1, ..., r$, *let* ψ_i *be an irreducible character of* G_i *such that* $\psi_i(-1) = -\psi_i(1)$ *and* $\mathbf{K}(\psi_i|_{H_i}) = \mathbf{K}$. *Let* $M_i \neq 0$ *be a* ψ_i*-quasihomogeneous* $\mathbf{K}G_i$*-module and let* χ_i *be the character afforded by* M_i. *Then, for* $g_i \in G_i$,

$$(\chi_1 \vee \cdots \vee \chi_r)(g_1, \cdots, g_r) = \begin{cases} 0 & \text{if } g_i \notin H_i, \text{ for some } i, \\ 2^{r-1}\chi_1(g_1)\cdots\chi_r(g_r) & \text{if } g_i \in H_i, \text{ for all } i. \end{cases}$$

Furthermore, there is some irreducible $(G_1 \vee \cdots \vee G_r)$*-character* ψ *with the property that* $M_1 \vee \cdots \vee M_r$ *is* ψ*-quasihomogeneous. For any such* ψ, $\psi(-1) = -\psi(1)$ *and* $\psi(g) \in \mathbf{K}$ *whenever* g *is in the distinguished subgroup of index two of* $G_1 \vee \cdots \vee G_r$. *In addition,*

$$[[\psi]] = [[\psi_1]]\hat{\otimes}\cdots\hat{\otimes}[[\psi_r]]\hat{\otimes}(\mathbf{K}[i])^{r-1}.$$

Proof: This is just Theorem 4.3 of Turull [25]. ∎

The above corollary associates to the collection of irreducible characters $\psi_1, \ldots, \psi_r$ an irreducible character ψ defined up to multiplication by the irreducible character of $G_1 \vee \cdots \vee G_r$ whose kernel is the distinguished subgroup of index two, and gives a formula for $[[\psi]]$. This formula suggests that we define a new operation $\vee$ on BW($\mathbf{K}$) by $A \vee B = A\hat{\otimes}B\hat{\otimes}\mathbf{K}[i]$ for $A, B \in BW(\mathbf{K})$. With this notation, $[[\psi]] = [[\psi_1]] \vee \cdots \vee [[\psi_r]]$.

Proposition 2.7.4 *The operation* $\vee$ *yields as abelian group structure on* $BW(\mathbf{K})$. *The map*

$$\varphi : \big(BW(\mathbf{K}), \hat{\otimes}\big) \to (BW(\mathbf{K}), \vee)$$

defined by

$$\varphi(A) = A\hat{\otimes}\mathbf{K}[j],$$

where $\mathbf{K}[j]$ *is the two-dimensional graded algebra with* $(\mathbf{K}[j])_1 = \mathbf{K}j$ *and* $j^2 = 1$ *is a group isomorphism. Furthermore, when transported to triples, the* $\vee$ *operation satisfies the following, where* $D, E \in Br(\mathbf{K})$ *and* $a, b \in (\mathbf{K} \setminus \{0\})/(\mathbf{K} \setminus \{0\})^2$*:*

(1) $[D, 1, a] \vee [E, 1, b] = [DE(a, b), 1, ab]$.

(2) $[D, 0, a] \vee [E, 0, b] = [DE(a, b), 1, -ab]$.

(3) $[D, 0, a] \vee [E, 1, b] = [DE(-a, b), 0, ab]$.

(4) $[1, 1, 1]$ *is the identity for* $\vee$.

(5) $[D, 0, a]^{-1} = [D^{-1}, 0, -a]$ (inverse under $\vee$).

(6) $[D, 1, a]^{-1} = [D^{-1}(a, -1), 1, a]$ (inverse under $\vee$).

Proof: The class in $BW(\mathbf{K})$ of the central simple graded algebra $\mathbf{K}[j]$ is described by the triple $[1, 1, 1]$, and that of $\mathbf{K}[i]$ is described by $[1, 1, -1]$. Now by Theorem 2.5.4,

$$[1, 1, 1][1, 1, -1] = [1, 0, 1],$$

and $[1, 0, 1]$ represents the identity of $(BW(\mathbf{K}), \hat{\otimes})$. Hence, φ is an isomorphism. The formulas (1) through (6) follow from this and the corresponding ones in Theorem 2.5.4, together with the definition of $\vee$. ∎

2.8 Fundamental Algebraic Results

In this section, we quote some algebraic results which will be used throughout the dissertation.

Theorem 2.8.1 *If* A, B *and* C *are* $\mathbf{K}$*-algebras, then* $B \otimes C \simeq A$ *if and only if* A *contains subalgebras* B' *and* C' *such that*

(i) $B' \simeq B$ *and* $C' \simeq C$ *as* $\mathbf{K}$*-algebras,*

(ii) $C' \subseteq C_A(B)$*, and*

(iii) there exists basis $\{x_i : i \in I\}$ *of* B' *and* $\{y_j : j \in J\}$ *of* C' *such that* $\{x_i y_j : (i, j) \in I \times J\}$ *is a basis of* A.

If A *is finite dimensional, then (iii) can be replaced by*

(iv) A is generated as an $\mathbf{K}$*-algebra by* $B' \cup C'$ *and* $dimA = (dimB)(dimC)$.

Proof: See Proposition C in Section 9.2 of Pierce [17]. ■

Theorem 2.8.2 *Let* B *and* C *be subalgebras of the finite dimensional* $\mathbf{K}$*-algebra* A *such that* $C \subseteq C_A(B)$ *and* B *is a central simple. The following conditions are equivalent.*

(i) $A = BC$.

(ii) $dim_{\mathbf{K}}A = (dim_{\mathbf{K}}B)(dim_{\mathbf{K}}C)$.

(iii) The inclusion mappings of B *and* C *into* A *induce an isomorphism*

$$B \otimes C \simeq A,$$

here BC *denotes the product of two subalgebras of* A.

Proof: See Proposition a in Section 12.4 of Pierce [17]. ■

Theorem 2.8.3 *Let* B *be a simple subalgebra of the finite dimensional central simple* $\mathbf{K}$*-algebra* A*. Then*

(i) $C_A(B)$ *is simple.*

(ii) $(dim_{\mathbf{K}}B)(dim_{\mathbf{K}}C_A(B)) = dim_{\mathbf{K}}A$.

(iii) $C_A(C_A(B)) = B$.

(iv) If B *is central simple, then* $C_A(B)$ *is central simple, and* $A = B \otimes C_A(B)$.

Proof: See the theorem on page 232 of Pierce [17]. ■

Theorem 2.8.4 *Assume* $\mathbf{R}$ *is a commutative ring with unity 1. Let* A *be a right or left semi-simple Artinian* $\mathbf{R}$*-algebra.*

(i) There exist natural numbers $n_1, n_2, ..., n_r$ *and* $\mathbf{R}$*-division algebras* $D_1, D_2, ..., D_r$ *such that*

$$A \simeq M_{n_1}(D_1) \oplus M_{n_2}(D_2) \oplus ... \oplus M_{n_r}(D_r).$$

(ii) The pairs $(n_1, D_1), ..., (n_r, D_r)$ in (i) are uniquely determined (up to isomorphism) by A.

(iii) Conversely, if $n_1, n_2, ..., n_r \in N$ and $D_1, D_2, ..., D_r$ are division algebras over $\mathbf{R}$*, then $M_{n_1}(D_1) \oplus M_{n_2}(D_2) \oplus ... \oplus M_{n_r}(D_r)$ is a right or left semi-simple* $\mathbf{R}$*-algebra.*

Proof: See the Theorem in Section 3.5 of Pierce [17]. ■

Corollary 2.8.5 *Let $A = \langle x, y\rangle$ be a 4-dimensional* $\mathbf{K}$*-algebra, where* $\mathbf{K}$ *is a field of characteristic zero, generated by x and y such that*

$$xy = yx, \; x^2 = a \in \mathbf{K}^\times, \; y^2 = b \in \mathbf{K}^\times.$$

Here A is a direct sum of fields. Furthermore, let $\mathbf{F}$ *be any one of the summands, then*

$$\mathbf{F} \simeq \mathbf{K}\left(\sqrt{a}, \sqrt{b}\right).$$

Proof: By Theorem 2.8.1, we get

$$A \simeq < x > \otimes < y > .$$

Note

$$< x > \simeq \begin{cases} \mathbf{Q}\left(\sqrt{x^2}\right) & \text{if } x^2 \text{ is not a square of } \mathbf{K}^\times \\ \mathbf{Q} \oplus \mathbf{Q} & \text{otherwise.} \end{cases}$$

Similarly

$$< y > \simeq \begin{cases} \mathbf{Q}\left(\sqrt{y^2}\right) & \text{if } y^2 \text{ is not a square of } \mathbf{K}^\times \\ \mathbf{Q} \oplus \mathbf{Q} & \text{otherwise.} \end{cases}$$

We conclude that A is a direct sum of fields. Therefore, A is semisimple.

Let $\mathbf{F}$ be one of the summands of A. Then $\mathbf{F}$ is generated by the image of the projection map from A to $\mathbf{F}$. It follows immediately

$$\mathbf{F} \simeq \mathbf{K}\left(\sqrt{a}, \sqrt{b}\right).$$

The proof is now complete. ■

CHAPTER 3
REPRESENTATIONS

In this chapter, we define and construct some basic spin representations including $X^{\lambda,\mu}$, $\Theta^{\varepsilon_1,\varepsilon_2}$, $\Phi^{\lambda,\mu}$, $\rho^{\varepsilon_1,\varepsilon_2}$, $\Psi^{\varepsilon_1,\varepsilon_2}$. We also quote some results from the ordinary representations of W_n and the projective representations of Symmetric groups. We show in Chapter Four that all the spin representations of W_n can be constructed from these basic spin representations. Finally, in this chapter, we study the structures of the algebras associated with the twisted product of two or three modules. These results are directly applied in Chapter Four and greatly simplify the calculation of the algebras associated with the irreducible spin representations of the double covers of the group W_n.

3.1 The Ordinary Representations of W_n

In this section, we will recall the definition of the irreducible representation $X^{\lambda,\mu}$ of W_n and quote some basic properties related to $X^{\lambda,\mu}$.

For each partition λ of n, let X^λ denote the irreducible representation of S_n indexed by λ, and let χ^λ denote the corresponding character. The irreducible representations of W_n are indexed by ordered pairs of partitions (λ, μ) with $|\lambda|+|\mu| = n$ (e.g., see Zelevinski [30]). We will write $X^{\lambda,\mu}$ for the representation and $\chi^{\lambda,\mu}$ for the character indexed by the pair (λ, μ). In the usual parameterization, $X^{\lambda,\emptyset}$ is the extension of X^λ from S_n to W_n in which the short reflections t_i act trivially, and $X^{\emptyset,\lambda}$ is the product δX^λ. In the general case, assuming $|\lambda| = k$ and $|\mu| = n-k$, one defines

$$X^{\lambda,\mu} = X^{\lambda,\emptyset} \circ X^{\emptyset,\mu}$$

where $\circ$ denotes induction from $W_k \times W_{n-k}$ to W_n; i.e.,

$$V_1 \circ V_2 = (V_1 \otimes V_2) \uparrow W_n,$$

for any $\mathbf{C}W_k$-module V_1 and $\mathbf{C}W_{n-k}$-module V_2. We will also use $\circ$ to denote the corresponding operation on characters, so that $\chi^{\lambda,\mu} = \chi^{\lambda,\emptyset} \circ \chi^{\emptyset,\mu}$.

In dealing with some cases of Chapter Four, we will need to consider $X^{\lambda,\mu}$ and their associators with respect to each of four linear characters of W_n.

First, we have the following lemma, which follows from p.419 - p.422 of Stembridge [23].

Lemma 3.1.1 *$\chi^{\lambda,\mu}$ has the following relations with the linear representations:*

(1) $\varepsilon\chi^{\lambda,\mu} = \chi^{\lambda,\mu}$ iff $\lambda, \mu \in SC$,

(2) $\delta\chi^{\lambda,\mu} = \chi^{\lambda,\mu}$ iff $\lambda = \mu$,

(3) $\varepsilon\delta\chi^{\lambda,\mu} = \chi^{\lambda,\mu}$ iff $\lambda = \mu'$,

(4) $\varepsilon\chi^{\lambda,\mu} = \chi^{\lambda,\mu}$ and $\delta\chi^{\lambda,\mu} = \chi^{\lambda,\mu}$ iff $\lambda = \mu \in SC$.

Second, we consider the problem of constructing the associators of representations $X^{\lambda,\mu}$ of W_n. It will be convenient in what follows to have an explicit description of a module for $X^{\lambda,\mu}$ in terms of modules for X^{λ} and X^{μ}. For this, we need to specify a particular embedding of $W_k \times W_{n-k}$ in W_n; the most obvious choice is the inverse image $W_{k,n-k}$ of the Young subgroup $S_{k,n-k}$ in W_n. Given $\omega_1 \in W_k$ and $\omega_2 \in W_{n-k}$, we will identify (ω_1, ω_2) with the corresponding element of $W_{k,n-k}$. As a collection of (left) coset representatives for $W_{k,n-k}$ in W_n, we will use the set $W^k \subset S_n$ consisting of all permutations ω of $1, 2, \cdots, n$ such that $\omega(1) < \cdots < \omega(k)$ and $\omega(k+1) < \cdots < \omega(n)$. Now given modules V_1 and V_2 for $\mathbf{C}W_k$ and $\mathbf{C}W_{n-k}$ with characters $\chi^{\lambda,\emptyset}$ and $\chi^{\emptyset,\mu}$, we may impose the module structure of $\chi^{\lambda,\mu}$ on the vector space $\mathbf{C}W^k \otimes V_1 \otimes V_2$ by defining

$$\omega(\omega_0 \otimes v_1 \otimes v_2) = \delta(\omega_2)\omega_0' \otimes \omega_1 v_1 \otimes \omega_2 v_2 \tag{3.1}$$

for all $v_i \in V_i$, where ω_0, $\omega_0' \in W^k$, $\omega\omega_0 = \omega_0'(\omega_1, \omega_2)$, and $(\omega_1, \omega_2) \in W_{(k,n-k)}$.

In this dissertation, as we said before, we are interested in the Schur index over $\mathbf{Q}$, therefore, we need to find the $\mathbf{Q}$-associators for each of the linear characters of W_n.

It is well-known that X^λ can be realized over $\mathbf{Q}$. Therefore, we can assume that V_1 and V_2 are a $\mathbf{Q}W_k$-module and a $\mathbf{Q}W_{n-k}$- module respectively, and $X^{\lambda,\mu}$ can be realized over $\mathbf{Q}$ by imposing the module structure of $X^{\lambda,\mu}$ on the vector space $\mathbf{Q}W^k \otimes V_1 \otimes V_2$ by the above definition.

• The ε-associator.

Let us assume henceforth that $\lambda, \mu \in SC$, and let S_1 and S_2 denote ε-associators (over $\mathbf{Q}$) for V_1 and V_2 as symmetric group representations. Define

$$S(\omega_0 \otimes v_1 \otimes v_2) = \varepsilon(\omega_0)\omega_0 \otimes S_1 v_1 \otimes S_2 v_2, \tag{3.2}$$

for $\omega_0 \in W^k$ and $v_i \in V_i$.

Lemma 3.1.2 *S is an ε-associator of $X^{\lambda,\mu}$, and $S^2 = S_1^2 S_2^2$.*

Proof: It follows from the proof of Theorem 6.2 of Stembridge [23]. ■

Theorem 3.1.3 *Let $\lambda \in SC$ be a partition of n and S be an ε-associator of X^λ, then*

$$\mathbf{Q}\left(\sqrt{S^2}\right) = \mathbf{Q}\left(\sqrt{(-1)^{\frac{n-k}{2}} \prod_i^k h_{ii}^\lambda}\right),$$

where k denotes the length of the main diagonal of the Young diagram of λ, and h_{ii}^λ are the main hooks of λ (refer to James and Kerber [10]).

Proof: Let N be a $\mathbf{Q}S_n$-module affording X^λ. Consider the group pair (S_n, A_n). Since $\lambda = \lambda'$, the graded algebra $A = A_{A_n}(N)$ is a **CSGA** of odd type. Let

$$\begin{aligned} A &\simeq [D, 1, d], A_0 = A_{S_n}(N), \\ Z(A) &\simeq \mathbf{Q} + i\mathbf{Q}, i^2 = d \in \mathbf{Q}^\times, \end{aligned}$$

and note i is an ε-associator.

Then it follows from 2.5.13 of James and Kerber [10] that

$$\mathbf{Q}(\sqrt{d}) = \mathbf{Q}\left(\sqrt{(-1)^{\frac{n-k}{2}} \prod_{i}^{k} h_{ii}^{\lambda}}\right)$$

.

Since all other ε-associators are rational multiples of i, we get

$$\mathbf{Q}\left(\sqrt{S^2}\right) = \mathbf{Q}\left(\sqrt{i^2}\right) = \mathbf{Q}\left(\sqrt{d}\right).$$

The proof is complete. ∎

Definition 3.1.4 *We define*

$$Z_\lambda^* = (-1)^{\frac{n-k}{2}} \prod_{i}^{k} h_{ii}^{\lambda}.$$

Remark: We will use notation Z_λ^* throughout the dissertation. Therefore, if S is an ε-associator of $X^{\lambda,\mu}$, then $S^2 = Z_\lambda^* Z_\mu^*$.

• δ-associator.

Assume $\lambda = \mu$. Then λ is a fixed partition of $k = n/2$, and we assume $V_1 = V_2 = V$. Denote by $u \in S_n$ the involution $(1, k+1)(2, k+2) \cdots (k, 2k)$. Define

$$T(\omega_0 \otimes v_1 \otimes v_2) = \omega_0 u \otimes v_2 \otimes v_1,$$

for $\omega_0 \in W^{\frac{n}{2}}$ and $v_i \in V$.

Lemma 3.1.5 *T is a δ-associator of $X^{\lambda,\lambda}$, and $T^2 = 1$.*

Proof: It follows from the proof of Theorem 6.3 of Stembridge [23]. ∎

• $\varepsilon\delta$-associator.

Assume $\lambda = \mu'$. We assume that λ is a fixed partition of $k = n/2$, and let $V = V_1$ be a module with character $\chi^{\lambda,\emptyset}$. To impose the module structure of $X^{\lambda,\lambda'}$

on $\mathbf{Q}W^{n/2} \otimes V \otimes V$ we need to modify (3.1) to take into account the fact that $V_2 = \varepsilon \otimes V_1$. In these terms, the action of $\omega \in W_n$ can be realized via

$$\omega(\omega_0 \otimes v_1 \otimes v_2) = \varepsilon\delta(\omega_2)\omega_0' \otimes \omega_1 v_1 \otimes \omega_2 v_2,$$

where $\omega\omega_0 = \omega_0'(\omega_1, \omega_2)$, as usual. Define

$$U(\omega_0 \otimes v_1 \otimes v_2) = \varepsilon\delta(\omega_0)\omega_0 u \otimes v_2 \otimes v_1,$$

where u is the involution defined before.

Lemma 3.1.6 *U is $\varepsilon\delta$-associator of $X^{\lambda,\lambda'}$ and $U^2 = (-1)^{n/2}$.*

Proof: It follows from the proof of Theorem 6.4 of Stembridge [23]. ■

In the study of spin characters of W_n, sometimes we need to consider the representation $X^{\lambda,\mu}$ in the situation that $X^{\lambda,\mu}$ is self-associate with both linear representations ε and δ. So an ε-associator S and a δ-associator T exists in that situation. This forces us to find the relations between S and T.

Observe that $X^{\lambda,\mu}$ is self-associate with respect to both ε and δ if and only if $\lambda = \mu \in SC$. In that case, we have

$$ST(\omega_0 \otimes v_1 \otimes v_2) = \varepsilon(\omega_0 u)\omega_0 u \otimes S_1 v_2 \otimes S_2 v_1,$$

where $S_1 = S_2$ denotes an ε-associator of $V = V_1 = V_2$.

Theorem 3.1.7 *S and T have the following relations:*

(1) S, T commute iff $n/2$ is even iff $n \equiv 0$ (mod 4).

(2) S, T anti-commute iff $n/2$ is odd iff $n \equiv 2$ (mod 4).

Proof: It follows from the definitions of S and T that

$$TS(\omega_0 \otimes v_1 \otimes v_2) = \varepsilon(\omega_0)\omega_0 u \otimes S_1 v_2 \otimes S_2 v_1,$$

this implies $ST = \varepsilon(u)TS$.

Since $\varepsilon(u) = (-1)^{n/2}$, we conclude that S and T commute when $n/2$ is even and anti-commute when $n/2$ is odd. Therefore the theorem follows. ■

Remark: The representations $X^{\lambda,\mu}$ yield spin representations of the double covers $W_n[1,1,1,1,1]$. We will use the same notation for these spin representations in the rest of this dissertation.

3.2 The Projective Representations of S_n

The projective representations of W_n with factor set $[\varepsilon_1, \varepsilon_2, -1, 1, 1]$ have a close relation with those of symmetric groups. We will detail this in Chapter Four. As preliminaries, we quote some well-known results about projective representations of symmetric groups in this section. These are mainly taken from Turull [25], Stembridge [22] and [23].

Define $\tilde{S}_n$ to be the group generated by $-1, \sigma_1, \cdots, \sigma_{n-1}$ such that

$$(-1)^2 = 1, \qquad (-1)\sigma_j = \sigma_j(-1), \qquad \sigma_j^2 = 1,$$
$$(\sigma_j\sigma_k)^2 = -1 \ (|j-k| \geq 2), \quad (\sigma_j\sigma_{j+1})^3 = 1.$$

For each $\lambda \in DP_n^+$ one associates a complex irreducible character of $\tilde{S}_n$, namely φ^λ. For each $\lambda \in DP_n^-$ one associates a pair of complex irreducible characters of $\lambda \in DP_n^+$, namely φ_+^λ and φ_-^λ; see Schur [20] or Stembridge [22] for the definition.

Theorem 3.2.1 *The characters φ^λ for $\lambda \in DP_n^+$ and $\varphi_\pm^\lambda$ for $\lambda \in DP_n^-$ are precisely the irreducible spin characters of $\tilde{S}_n$.*

If λ is a partition of n, we denote by $\tilde{S}_\lambda$ the subgroup of $\tilde{S}_n$ generated by -1 and σ_j for $j \in \{1, \cdots, n\} \setminus \{\lambda_1, \lambda_1 + \lambda_2, \cdots, \lambda_1 + \lambda_2 + \cdots + \lambda_l\}$. Then $\tilde{S}_\lambda$ has natural copies of $\tilde{S}_{\lambda_1}, \cdots, \tilde{S}_{\lambda_l}$ as normal subgroups. One can construct irreducible characters of $\tilde{S}_\lambda$, namely θ^λ if λ is even and θ_+^λ and θ_-^λ if λ is odd, see Stembridge [22] for details.

Consider the group pair $(\tilde{S}_n, \tilde{A}_n)$, for each irreducible character φ^λ of $\tilde{S}_n$ we have $[[\varphi^\lambda]]$ (see Corollary 2.6.9). This is described by a triple (see Theorem 2.5.4). In Turull [25], this triple is computed using the operation described in Proposition 2.7.4. Notice that in our case, the ε of Stembridge [23] is actually -1.

Definition 3.2.2 *For each positive integer n, we define a* **CSGA** *as follows.*

$$[n] = \left[(-1,-1)^{\binom{n+1}{4}} \left(-2(-1)^{\frac{n+1}{2}}, (-1)^{\frac{n-1}{2}}n\right), 1, n(-1)^{\frac{n-1}{2}}\right] \text{ if } n \text{ is odd.}$$

and

$$[n] = \left[(-1,-1)^{\binom{n+1}{4}} \left((-1)^{\frac{n}{2}}, n\right), 0, -2(-1)^{\frac{n}{2}}n\right] \text{ if } n \text{ is even.}$$

Theorem 3.2.3 *With the notation $[n]$, we have*

$$[[\varphi^\lambda]] = [[\theta^\lambda]] = [\lambda_1] \vee \cdots \vee [\lambda_m].$$

Proof: This is just Theorem 6.2 in Turull [25]. ∎

Let ε be the "sign" homomorphism. We have the following.

Corollary 3.2.4 *φ^λ has the following relations with respect to ε.*

(1) $\varepsilon\varphi^\lambda = \varphi^\lambda$.

(2) $\varepsilon\varphi^\lambda_+ = \varphi^\lambda_-$.

Corollary 3.2.5 *If $\lambda \in DP^+$, then $2\varphi^\lambda$ can be realized over* $\mathbf{Q}$.

If $\lambda \in DP^-$, then $2(\varphi^\lambda_+ + \varphi^\lambda_-)$ can be realized over $\mathbf{Q}$.

Proof: This follows from above theorem and the fact that the Schur index of each spin character of $\tilde{S}_n$ is at most 2 (Turul [24]). ∎

3.3 The Definition of $\Theta^{\varepsilon_1,\varepsilon_2}$

Any group G with a $\mathbf{Z}_2^2$-quotient has a two-dimensional projective representation arising from the fact that the dihedral group of order 8 doubly covers $\mathbf{Z}_2^2$. To be

more precise, let $\Theta_0 : \mathbf{Z}_2^2 \longrightarrow PGL_2(\mathbf{Q})$ denote the projective representation obtained from the reflection representation of the dihedral group modulo its center. We may thus obtain a projective G-representation, also to be denoted by Θ_0, via

$$G \longrightarrow G/H \longrightarrow \mathbf{Z}_2^2 \longrightarrow GL_2(\mathbf{Q}).$$

The following theorem gives details about the construction of this projective representation.

Theorem 3.3.1 *Let $\alpha = [\varepsilon_1, \varepsilon_2, 1, -1, 1]$, then the double cover $W_n[\alpha]$ has a 2-dimensional irreducible spin representation $\Theta_0^{\varepsilon_1,\varepsilon_2}$ such that $\Theta_0^{\varepsilon_1,\varepsilon_2}$ can be realized over $\mathbf{Q}$ except the case $\varepsilon_1 = -1 = \varepsilon_2$, and in that situation, $2\Theta_0^{\varepsilon_1,\varepsilon_2}$ can be realized over $\mathbf{Q}$. Let $\theta_0^{\varepsilon_1,\varepsilon_2}$ denote the character afforded by $\Theta_0^{\varepsilon_1,\varepsilon_2}$, then $\theta_0^{\varepsilon_1,\varepsilon_2}$ has the following character values:*

$$\theta_0^{\varepsilon_1,\varepsilon_2}(\nu, w) = \begin{cases} 2\nu & \text{if } w \in W_n' \\ 0 & \text{if } w \notin W_n' \end{cases}$$

where (ν, w) is any element of $W_n[\varepsilon_1, \varepsilon_2, 1, -1, 1]$, namely, $\nu \in \{-1, 1\}$ and $w \in W_n$.

Proof: We divide the proof into several cases. For each case we choose special 2×2-matrices x_0 and y_0 such that the representation is achieved via

$$\begin{array}{ccc} W_n[\alpha] & \to & GL_2(\mathbf{Q}) \\ \sigma_i & \longmapsto & x_0 \\ \tau & \longmapsto & y_0. \end{array}$$

If $\alpha = [1, 1, 1, -1, 1]$, we can take

$$x_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad y_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

notice $x_0^2 = 1 = y_0^2, x_0 y_0 = -y_0 x_0$, and the group generated by x_0 and y_0 is a dihedral group of order 8.

If $\alpha = [-1, 1, 1, -1, 1]$, we can take

$$x_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad y_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

notice $x_0^2 = -1, y_0^2 = 1, x_0 y_0 = -y_0 x_0$, and the group generated by x_0 and y_0 is a dihedral group of order 8.

If $\alpha = [1, -1, 1, -1, 1]$, we can take

$$x_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad y_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

notice $x_0^2 = 1, y_0^2 = -1, x_0 y_0 = -y_0 x_0$, and the group generated by x_0 and y_0 is a dihedral group of order 8.

If $\alpha = [-1, -1, 1, -1, 1]$, we can take

$$x_0 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad y_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

notice $x_0^2 = -1 = y_0^2, x_0 y_0 = -y_0 x_0$, and the group generated by x_0 and y_0 is a quaternion group of order 8.

It is easy to check that $\Theta_0^{-1,-1}$ can not be realized over $\mathbf{Q}$. We show that $2\Theta_0^{-1,-1}$ can be realized over $\mathbf{Q}$.

Let $\alpha = [-1, -1, 1, -1, 1]$, we can take

$$x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Then $x^2 = -1$, $y^2 = -1$, $xy = -yx$, and the group generated by x and y is a quaternion group of order 8.

Denote by $\Theta^{-1,-1}$ the representation associated with x and y. It is straightforward to check $\Theta^{-1,-1} = 2\Theta_0^{-1,-1}$.

From the above construction of each 2-dimensional representation, the character values of $\theta_0^{\varepsilon_1,\varepsilon_2}$ can be very easily verified.

Then the proof is finished. $\blacksquare$

Remark : To simplify the notations, we will use Θ_0 instead of $\Theta_0^{1,1}$ because this representation will be used quite frequently. This simplification is also applied to their corresponding characters. Namely, we will use θ_0 to denote the character of Θ_0. We will keep all these notations throughout the dissertation unless otherwise mentioned.

Theorem 3.3.2 *Let* $\alpha = [\varepsilon_1, \varepsilon_2, 1, -1, 1]$, *then the double cover* $W_n[\alpha]$ *has a four dimensional spin representation* $\Theta^{\varepsilon_1,\varepsilon_2}$ *such that* $\Theta^{\varepsilon_1,\varepsilon_2}$ *can be realized over* $\mathbf{Q}$. *We also use* $\theta^{\varepsilon_1,\varepsilon_2}$ *to denote the character associated with* $\Theta^{\varepsilon_1,\varepsilon_2}$.

Proof: This is just a corollary of the above theorem. But for later use, we give an explicit realization for each representation.

We can take $\Theta^{\varepsilon_1,\varepsilon_2} = 2\Theta_0^{\varepsilon_1,\varepsilon_2}$. To be more precise, we consider the four different cases.

For $\alpha = [1, 1, 1, -1, 1]$, we can take

$$x = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Then $x^2 = 1,\ y^2 = 1,\ xy = -yx$, and the group generated by x and y is a dihedral group of order 8.

If $\alpha = [-1, 1, 1, -1, 1]$, we can take

$$x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Then $x^2 = -1,\ y^2 = 1,\ xy = -yx$, and the group generated by x and y is a dihedral group of order 8.

If $\alpha = [1, -1, 1, -1, 1]$, we can take

$$x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Then $x^2 = 1,\ y^2 = -1,\ xy = -yx$, and the group generated by x and y is a dihedral group of order 8.

Finally, if $\alpha = [-1, -1, 1, -1, 1]$, the above theorem already actually provided an explicit realization.

Now the proof is complete. ■

Proposition 3.3.3 *Let* $\alpha = [\varepsilon_1, \varepsilon_2, 1, -1, 1]$. *Following the above notation, let* O *be a* $\mathbf{Q}W_n[\alpha]$*-module affording* $\Theta^{\varepsilon_1, \varepsilon_2}$, *then we have*

(i) $\Theta^{\varepsilon_1,\varepsilon_2} = 2\Theta_0^{\varepsilon_1,\varepsilon_2}$.

(ii) $\Theta^{\varepsilon_1,\varepsilon_2}$ *is self-associate with each of linear characters* ε, δ, $\varepsilon\delta$ *and the associators can be chosen to be* y, x, *and* xy, *respectively.*

(iii)$A_{W_n[\alpha]}(O) := End_{\mathbf{Q}W_n[\alpha]}(O) \simeq (\varepsilon_1, \varepsilon_2)$ *where* $(\varepsilon_1, \varepsilon_2)$ *denotes the quaternion algebra generated by* a, b *such that* $a^2 = \varepsilon_1, b^2 = \varepsilon_2$ *and* $ab = -ba$.

(iv) $[x,\ A_{W_n[\alpha]}(O)] = 1, [y,\ A_{W_n[\alpha]}(O)] = 1$. *Here the notation means that* x, y *can commute with any element of* $A_{W_n[\alpha]}(O)$.

Proof: We first prove (i). Notice that the dihedral group of order 8 and the quaternion group of order 8 have only one irreducible representation of degree 2, then checking is very straightforward.

Checking that y, x, and xy can be chosen to be corresponding associators is also straightforward. Now let's prove (iii).

For $\alpha = [1, 1, 1, -1, 1]$, the direct calculation shows that $End_{\mathbf{Q}W_n[\alpha]}(O)$ has dimension 4 and consists of all the following matrices:

$$\begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix}$$

where $a, b, c, d \in \mathbf{Q}$.

Let

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

then $A^2 = 1, B^2 = 1, AB = -BA$ and

$$End_{\mathbf{Q}W_n[\alpha]}(O) =< A, B >\simeq (A^2, B^2) \simeq (\varepsilon_1, \varepsilon_2).$$

For $\alpha = [-1, 1, 1, -1, 1]$, the calculation shows that $End_{\mathbf{Q}W_n[\alpha]}(O)$ has dimension 4 and consists of all the following matrices:

$$\begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix}$$

where $a, b, c, d \in \mathbf{Q}$.

Let

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

then $A^2 = -1, B^2 = 1, AB = -BA$ and

$$End_{\mathbf{Q}W_n[\alpha]}(O) =< A, B >\simeq (A^2, B^2) \simeq (\varepsilon_1, \varepsilon_2).$$

For $\alpha = [-1, 1, 1, -1, 1]$, the calculation shows that $End_{\mathbf{Q}W_n[\alpha]}(O)$ has dimension 4 and consists of all the following matrices:

$$\begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix}$$

where $a, b, c, d \in \mathbf{Q}$.

Let

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

then $A^2 = 1, B^2 = -1, AB = -BA$ and

$$End_{\mathbf{Q}W_n[\alpha]}(O) =< A, B >\simeq (A^2, B^2) \simeq (\varepsilon_1, \varepsilon_2).$$

For $\alpha = [-1, -1, 1, -1, 1]$, the calculation shows that $End_{\mathbf{Q}W_n[\alpha]}(O)$ has dimension 4 and consists of all the following matrices:

$$\begin{pmatrix} a & b & c & d \\ -b & a & d & -c \\ -c & -d & a & b \\ -d & c & -b & a \end{pmatrix}$$

where $a, b, c, d \in \mathbf{Q}$.

Let

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

then $A^2 = -1, B^2 = -1, AB = -BA$ and

$$End_{\mathbf{Q}W_n[\alpha]}(O) =< A, B >\simeq (A^2, B^2) \simeq (\varepsilon_1, \varepsilon_2).$$

The proof of (iii) is finished.

(iv) follows from the above construction of $A_{W_n[\alpha]}(O)$.

Thus the proof of Proposition is complete. ■

Proposition 3.3.4 *Let $\alpha = [\varepsilon_1, \varepsilon_2, 1, -1, 1]$ and $\Theta_0^{\varepsilon_1,\varepsilon_2}, \Theta^{\varepsilon_1,\varepsilon_2}$ be the representations of $W_n[\alpha]$ defined in Theorem 3.3.1 and Theorem 3.3.2. Let χ be an irreducible spin representation of the group $W_n[\beta]$, then*

(i) $||\Theta_0^{\varepsilon_1,\varepsilon_2}\chi||^2 = |L(\chi)|$.

(ii) $||\Theta^{\varepsilon_1,\varepsilon_2}\chi||^2 = 4|L(\chi)|$.

Here all products mean the twisted products.

Proof: (i) follows from the proof of Theorem 2.4.2. (ii) follows (i) and Proposition 3.3.3 (ii). ■

Recall that Theorem 2.4.2 gives the structures of the twisted product $N \otimes M$, where M is an irreducible $\mathbf{C}W_n[\beta]$-module and N is a $\mathbf{C}W_n[\alpha]$-module affording θ_0. Actually Theorem 2.4.2 is also true if we replace θ_0 by $\theta_0^{\varepsilon_1,\varepsilon_2}$. To make self-contained and easy to read, we give the following theorem.

Theorem 3.3.5 *Let M be an irreducible $\mathbf{C}W_n[\beta]$-module and N be a $\mathbf{C}W_n[\alpha]$-module affording $\theta_0^{\varepsilon_1,\varepsilon_2}$. Then*

(a) If $L_M = \{1\}$, then $N \otimes M$ is an irreducible $\mathbf{C}G^{\beta\alpha}$-module.

(b) If $L_M = \{1, \varepsilon\}$, then $N \otimes M$ is a direct sum of two irreducible, non-isomorphic $\mathbf{C}G^{\beta\alpha}$-modules (and similarly for $L_M = \{1, \delta\}$ and $L_M = \{1, \varepsilon\delta\}$).

(c) If $L_M = \{1, \varepsilon, \delta, \varepsilon\delta\}$, then let S denote the ε-associator, and let T denote the δ-associator for M.

(i) If $ST = TS$, then $N \otimes M$ is a direct sum two copies of one irreducible $\mathbf{C}G^{\beta\alpha}$-modules

(ii) If $ST = -TS$, then $N \otimes M$ is a direct sum of 4 irreducible, non-isomorphic $\mathbf{C}G^{\beta\alpha}$-modules.

Proof: The results follows from the proof of Theorem 2.4.2 and Proposition 3.3.4. ∎

We close this section by proving a simple result relating the character $\theta^{\varepsilon_1,\varepsilon_2}$ and group pairs which will be used frequently in Chapter Four.

Lemma 3.3.6 *Let $G = W_n[\varepsilon_1, \varepsilon_1, 1, -1, 1]$ and O be a $\mathbf{Q}G$-module affording $\Theta^{\varepsilon_1,\varepsilon_2}$. Let $H(\varepsilon)$ be the kernel of ε, $H(\delta)$ be the kernel of δ and $H(\varepsilon\delta)$ be the kernel of $\varepsilon\delta$. Then*

(i) $A_{H(\varepsilon)}(O) \simeq [(\varepsilon_1, \varepsilon_2), 1, \varepsilon_2]$ is a $\mathbf{CSGA}$ of odd type.

(ii) $A_{H(\delta)}(O) \simeq [(\varepsilon_1, \varepsilon_2), 1, \varepsilon_1]$ is a $\mathbf{CSGA}$ of odd type.

(iii) $A_{H(\varepsilon\delta)}(O) \simeq [(\varepsilon_1, \varepsilon_2), 1, -\varepsilon_1\varepsilon_2]$ is a $\mathbf{CSGA}$ of odd type.

Proof: We only prove (i). The proofs of (ii) and (iii) are similar.

Following the notations from the definition of $\Theta^{\varepsilon_1,\varepsilon_2}$, since $\Theta^{\varepsilon_1,\varepsilon_2}$ is ε-associate, $A_{H(\varepsilon)}(O)$ is a **CSGA** of odd type. Note y_0 is an ε-associator, and the result then follows from Proposition 3.3.3. ■

3.4 The Definition of $\rho^{\varepsilon_1,\varepsilon_2}$

This section is a continuation of the previous section. By considering the twisted product of Θ_0 and $\Theta_0^{\varepsilon_1,\varepsilon_2}$, we will define another special representation to be denoted by $\rho^{\varepsilon_1,\varepsilon_2}$.

Theorem 3.4.1 *Let $\alpha = [\varepsilon_1, \varepsilon_2, 1, 1, 1]$. Then the double cover $W_n[\alpha]$ has 8 linear characters. Four of which are $1, \varepsilon, \delta, \varepsilon\delta$, which send -1 to 1, and the other four are spin representations which send -1 to -1. Let Υ be any one of the linear spin representations, then the four linear spin representations are $\Upsilon, \varepsilon\Upsilon, \delta\Upsilon, \varepsilon\delta\Upsilon$. In addition, we have*

$$\Theta_0\Theta_0^{\varepsilon_1,\varepsilon_2} = \Upsilon + \varepsilon\Upsilon + \delta\Upsilon + \varepsilon\delta\Upsilon,$$

and

$$\Theta_0^{\varepsilon_1,1}\Theta_0^{1,\varepsilon_2} = \Upsilon + \varepsilon\Upsilon + \delta\Upsilon + \varepsilon\delta\Upsilon.$$

Proof: Since $W_n[\alpha]/W_n[\alpha]'$ has order 8, we conclude that $W_n[\alpha]$ has 8 linear characters.

By Theorem 3.3.5, $\Theta_0\Theta_0^{\varepsilon_1,\varepsilon_2}$ and $\Theta_0^{\varepsilon_1,1}\Theta_0^{1,\varepsilon_2}$ both are equal to the sum of four distinct linear spin representations. It is easy to check that $\Upsilon, \varepsilon\Upsilon, \delta\Upsilon, \varepsilon\delta\Upsilon$ are all linear spin representations. Therefore

$$\Theta_0\Theta_0^{\varepsilon_1,\varepsilon_2} = \Upsilon + \varepsilon\Upsilon + \delta\Upsilon + \varepsilon\delta\Upsilon$$

and

$$\Theta_0^{\varepsilon_1,1}\Theta_0^{1,\varepsilon_2} = \Upsilon + \varepsilon\Upsilon + \delta\Upsilon + \varepsilon\delta\Upsilon.$$

The theorem is now proved. ■

In the rest of dissertation, we will denote by $\rho^{\varepsilon_1,\varepsilon_2}$ the sum of the four linear spin representations of the group $W_n[\varepsilon_1, \varepsilon_2, 1, 1, 1]$, namely

$$\rho^{\varepsilon_1,\varepsilon_2} = \Upsilon + \varepsilon\Upsilon + \delta\Upsilon + \varepsilon\delta\Upsilon.$$

We will use $\varrho^{\varepsilon_1,\varepsilon_2}$ to denote the character afforded by $\rho^{\varepsilon_1,\varepsilon_2}$. These notations will be used throughout the dissertation without further mention.

Theorem 3.4.2 *Let $G = W_n[\varepsilon_1, \varepsilon_2, 1, 1, 1]$. Following the above notation, let $\rho^{\varepsilon_1,\varepsilon_2} = \Upsilon + \varepsilon\Upsilon + \delta\Upsilon + \varepsilon\delta\Upsilon$ be the sum of the four linear spin representations of G. Then $\rho^{\varepsilon_1,\varepsilon_2}$ can be afforded by a 4-dimensional $\mathbf{Q}G$-module M and furthermore we have*

(1) $End_{\mathbf{Q}G}(M) =< \mathcal{X}, \mathcal{Y} >$ such that $\mathcal{X}^2 = \varepsilon_1, \mathcal{Y}^2 = \varepsilon_2$ and $\mathcal{X}\mathcal{Y} = \mathcal{Y}\mathcal{X}$. In addition,

$$End_{\mathbf{Q}G}(M) \simeq \begin{cases} \mathbf{Q} \oplus \mathbf{Q} \oplus \mathbf{Q} \oplus \mathbf{Q} & \text{if } \varepsilon_1 = 1 = \varepsilon_2 \\ \mathbf{Q}(\sqrt{\varepsilon_1}, \sqrt{\varepsilon_2}) \oplus \mathbf{Q}(\sqrt{\varepsilon_1}, \sqrt{\varepsilon_2}) & \text{otherwise.} \end{cases}$$

(2) $\rho^{\varepsilon_1,\varepsilon_2}$ is self-associate with four linear representations $1, \varepsilon, \delta, \varepsilon\delta$ and there is an ε-associator $\mathcal{A}$ and a δ-associator $\mathcal{B}$ such that $\mathcal{A}^2 = 1, \mathcal{B}^2 = 1, \mathcal{A}\mathcal{B} = \mathcal{B}\mathcal{A}$ and $\mathcal{A}\mathcal{B}$ is an $\varepsilon\delta$-associator.

Furthermore, we have

$$\mathcal{A}\mathcal{X} = -\mathcal{X}\mathcal{A},\ \mathcal{A}\mathcal{Y} = \mathcal{Y}\mathcal{A},$$

$$\mathcal{B}\mathcal{X} = \mathcal{X}\mathcal{B},\ \mathcal{B}\mathcal{Y} = -\mathcal{Y}\mathcal{B}.$$

Proof: Let

$$\rho^{\varepsilon_1,\varepsilon_2} = \Upsilon + \varepsilon\Upsilon + \delta\Upsilon + \varepsilon\delta\Upsilon.$$

By Theorem 3.4.1, $\rho^{\varepsilon_1,\varepsilon_2} = \Theta_0^{\varepsilon_1,1}\Theta_0^{1,\varepsilon_2}$. Since both $\Theta_0^{\varepsilon_1,1}$ and $\Theta_0^{1,\varepsilon_2}$ can be realized over $\mathbf{Q}$ with 2-dimensional modules, we conclude that $\rho^{\varepsilon_1,\varepsilon_2}$ can be realized over $\mathbf{Q}$ with a 4-dimensional module.

Let V_1 be a $\mathbf{Q}W_n[\varepsilon_1, 1, 1, -1, 1]$-module affording $\Theta_0^{\varepsilon_1,1}$ and V_2 be a $\mathbf{Q}W_n[1, \varepsilon_2, 1, -1, 1]$-module affording $\Theta_0^{1,\varepsilon_2}$. The twisted tensor product $V_1 \otimes V_2$ affords the twisted product $\rho^{\varepsilon_1,\varepsilon_2} = \Theta_0^{\varepsilon_1,1}\Theta_0^{1,\varepsilon_2}$.

By Theorem 3.3.1, $\Theta_0^{\varepsilon_1,1}$ has an explicit representation

$$\Theta_0^{\varepsilon_1,1}(\sigma_i) = x, \Theta_0^{\varepsilon_1,1}(\tau) = a,$$

where $x^2 = \varepsilon_1, a^2 = 1, xa = -ax$.

Similarly, $\Theta_0^{1,\varepsilon_2}$ has an explicit representation

$$\Theta_0^{1,\varepsilon_2}(\sigma_i) = b, \Theta_0^{1,\varepsilon_2}(\tau) = y,$$

where $y^2 = \varepsilon_2, b^2 = 1, yb = -by$.

It is easy to check that $W_n[\varepsilon_1, \varepsilon_2, 1, 1, 1]$ has the following action on $V_1 \otimes V_2$

$$\sigma_i(v_1 \otimes v_2) = (x \otimes b)(v_1 \otimes v_2)$$

$$\tau(v_1 \otimes v_2) = (a \otimes y)(v_1 \otimes v_2).$$

Let $\mathcal{X} = x \otimes b, \mathcal{Y} = a \otimes y, \mathcal{A} = a \otimes 1, \mathcal{B} = 1 \otimes b$, then it is straightforward to check that $\mathcal{X}, \mathcal{Y}, \mathcal{A}$ and $\mathcal{B}$ satisfy all the requirements.

The structure of $End_{\mathbf{Q}G}(M)$ follows from the Corollary 2.8.5.

The proof is finished. ■

Remark : Denote $G = W_n[\varepsilon_1, \varepsilon_2, 1, 1, 1]$, then G/G' is an abelian group of order 8 and has a subgroup $H = \{-1, 1\}$ in its center. Let Θ be the linear spin representation of H, then Gallagher's Theorem (see 6.17 of Isaacs [9]) implies $\Theta^G = \Upsilon + \varepsilon\Upsilon + \delta\Upsilon + \varepsilon\delta\Upsilon$ where Υ is any one of linear spin representations of G/G'. All these representations can be viewed as linear spin representations of $G = W_n[\varepsilon_1, \varepsilon_2, 1, 1, 1]$. From this point of view, it is also clear that $\rho^{\varepsilon_1,\varepsilon_2}$ can be realized as a 4-dimensional $\mathbf{Q}G$-module.

Corollary 3.4.3 *With the above notations, we have*

(1) $||\rho^{\varepsilon_1,\varepsilon_2}\chi||^2 = 4|L(\chi)|$.

(2) $||\Theta_0^{\varepsilon_1,1}\Theta_0^{1,\varepsilon_2}\chi||^2 = 4|L(\chi)|$.

(3) $||\Theta_0\Theta^{\varepsilon_1,\varepsilon_2}\chi||^2 = 4^2|L(\chi)|$.

Proof: Notice

$$\rho^{\varepsilon_1,\varepsilon_2} = \Upsilon + \varepsilon\Upsilon + \delta\Upsilon + \varepsilon\delta\Upsilon$$

the (1) follows immediately.

(2) is exactly (1) since $\rho^{\varepsilon_1,\varepsilon_2} = \Theta_0^{\varepsilon_1,1}\Theta_0^{1,\varepsilon_2}$.

(3) follows from (2). ■

Theorem 3.4.4 *Let $\alpha = [\varepsilon_1, \varepsilon_2, 1, 1, 1]$ and Υ be one of the linear spin representation of $W_n[\alpha]$. Assume χ is an irreducible representation of $W_n[\beta]$, then*

$$(\Upsilon + \varepsilon\Upsilon + \delta\Upsilon + \varepsilon\delta\Upsilon)\chi = \begin{cases} \textit{sum of 4 distinct rep.} & \textit{if } |L(\chi)| = 1 \\ \textit{2(sum of 2 distinct rep.)} & \textit{if } |L(\chi)| = 2 \\ \textit{sum of 4 identical rep.} & \textit{if } |L(\chi)| = 4. \end{cases}$$

Proof: Since

$$\Theta_0\Theta_0^{\varepsilon_1,\varepsilon_2}\chi = (\Upsilon + \varepsilon\Upsilon + \delta\Upsilon + \varepsilon\delta\Upsilon)\chi$$

and

$$||\Theta_0\Theta_0^{\varepsilon_1,\varepsilon_2}\chi||^2_{W_n[\beta\alpha]} = 4|L(\chi)|,$$

then the result follows very easily. ■

3.5 The Definition of $\Phi^{\lambda,\mu}$

In this section, we will consider the group $W_n[1, 1, -1, 1, 1]$ and construct some special spin representations to be denoted by $\varphi^{\lambda,\mu}$. These special representations later can be used to characterize the spin representations for several double covers of W_n, including the spin representations of the group $W_n[\varepsilon_1, \varepsilon_2, -1, 1, 1]$.

We keep $\alpha = [1, 1, -1, 1, 1]$ in the following study unless otherwise particularly mentioned.

First we like to study the relations between the three groups $W_n[\alpha]$, $W_k[\alpha]$, and $W_{n-k}[\alpha]$.

We remark that $-1, \sigma_1, \sigma_2, \cdots, \sigma_{n-1}$ generate a subgroup of $W_n[\alpha]$ to be denoted by $\tilde{S_n}$ which is a double cover of S_n.

As is well known, the spin representations of $\tilde{S}_n$ are labeled by the partitions $\lambda \in DP$ of size n. We will use the notation Φ^λ for the representation indexed by λ and φ^λ for its character. A short description of Φ^λ is given in Section 3.2 and an explicit construction of the representations can be found in Nazarov [15] or in Turull [25] (in [25], $\tilde{S}_n$ corresponds to the double cover $\tilde{S}_n^{-1}$ of S_n). In the case $\lambda \in DP^-$ (i.e., $n - l(\lambda)$ is odd), there is actually a pair of ε-associate representations indexed by λ; we will write $\Phi^\lambda_\pm$ in situations where we need to emphasize that there are two associates. For $\lambda \in DP^+$, there is only one representation Φ^λ; it is self-associate with respect to ε.

In the following study, we will follow the notations of Stembridge [23] with a little modification. Let

$$W_n[\alpha] \to W_n \to S_n$$

be the natural mapping. Let $S_{k,n-k}$ be the Young subgroup of S_n and let $W_{k,n-k}$ be the preimage of $S_{k,n-k}$, finally let $W_{k,n-k}[\alpha]$ be the preimage of $W_{k,n-k}$.

Let $\tilde{S}_{k,n-k}$ be the preimage of the Young subgroup $S_{k,n-k}$ in the natural map $\tilde{S}_n \to S_n$. As we mentioned earlier $W_n[\alpha]$ contains a subgroup isomorphic to $\tilde{S}_n$. Similarly, $W_{k,n-k}[\alpha]$ contains a subgroup isomorphic to $\tilde{S}_{k,n-k}$.

Now $S_{k,n-k}$ has a $\mathbf{Z}_2 \times \mathbf{Z}_2$-quotient, so $S_{k,n-k}$ has a 2-dimensional projective representation. Equivalently, $\tilde{S}_{k,n-k}$ has a 2-dimensional spin representation to be denoted by Θ. The existence of this 2-dimensional spin representation results from the fact that the dihedral group of order 8 doubly covers $\mathbf{Z}_2 \times \mathbf{Z}_2$.

This Θ can be extended to $W_{k,n-k}[\alpha]$ by insisting the τ acts trivially. Let V_0 be a $\mathbf{Q}W_{k,n-k}[\alpha]$-module affording Θ. For any standard element $(\nu, w) \in W_{k,n-k}[\alpha]$ with $\nu \in \{-1, 1\}, w \in W_{k,n-k}$ and an element $v_0 \in V_0$, an explicit realization of Θ may be obtained by assigning

$$(\nu, w)v_0 = \nu(1, w)v_0$$

with the following properties:

$$\begin{aligned}(1,s_i)v_0 &= \begin{cases} x_0v_0 & \text{if } i<k \\ y_0v_0 & \text{if } i>k \end{cases} \\ (1,t)v_0 &= v_0\end{aligned}$$

where x_0 and y_0 denote any pair of anti-commuting involutions in $GL_2(\mathbf{Q})$, such as

$$x_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad y_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Given $w_1 \in W_k[\alpha]$ and $w_2 \in W_{n-k}[\alpha]$, we will use (w_1, w_2) to denote the element $w_1\hat{w}_2 \in W_{k,n-k}[\alpha]$ obtained by substituting σ_{j+k} for σ_j and τ_{j+k} for τ_j ($j = 1, 2, \ldots$) in any expression for w_2 in terms of the generators σ_j and τ_j of $W_{k,n-k}[\alpha]$.

Given $\lambda \in DP$, we define $\Phi^{\lambda,\emptyset}$ to be the representation obtained by extending Φ^λ from $\tilde{S}_n$ to $W_n[\alpha]$ by having the τ_j's act trivially, and we define $\Phi^{\emptyset,\lambda}$ to be the product $\delta\Phi^{\lambda,\emptyset}$. In the general case, given $\lambda, \mu \in DP$ with $|\lambda| = k$ and $|\mu| = n-k$, let V_1 be a $\mathbf{C}W_k[\alpha]$-module affording $\Phi^{\lambda,\emptyset}$, and V_2 be a $\mathbf{C}W_{n-k}[\alpha]$-module affording $\delta\Phi^{\mu,\emptyset}$. It follows that we may impose a $\mathbf{C}W_{k,n-k}[\alpha]$-module structure on $V_0 \otimes V_1 \otimes V_2$ via

$$(\nu, w)(v_0 \otimes v_1 \otimes v_2) = \nu\left((1,w)v_0 \otimes (1,w)v_1 \otimes (1,w)v_2\right)$$

with the following properties:

$$\begin{aligned}(1,s_j)(v_0 \otimes v_1 \otimes v_2) &= (1,s_j)v_0 \otimes (1,s_j)v_1 \otimes (1,s_j)v_2 \\ &= \begin{cases} x_0v_0 \otimes (1,s_j)v_1 \otimes v_2 & \text{if } j<k \\ y_0v_0 \otimes v_1 \otimes (1,s_j)v_2, & \text{if } j>k, \end{cases}\end{aligned}$$

and

$$\begin{aligned}(1,t_j)(v_0 \otimes v_1 \otimes v_2) &= (1,t_j)v_0 \otimes (1,t_j)v_1 \otimes (1,t_j)v_2 \\ &= \begin{cases} v_0 \otimes (1,t_j)v_1 \otimes v_2 & \text{if } j<k \\ y_0v_0 \otimes v_1 \otimes (1,t_j)v_2, & \text{if } j>k, \end{cases}\end{aligned}$$

where

$$x_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad y_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We define $V^{\lambda,\mu}$ to be any of the irreducible submodules of $V_0 \otimes V_1 \otimes V_2$. By Stembridge [23] (see page 413), there are essentially three possible types of structure:

Case 1. If neither V_1 nor V_2 is self-associate with respect to the sign character ε, then $V_0 \otimes V_1 \otimes V_2$ is irreducible, so in this case, we have $V_0 \otimes V_1 \otimes V_2 = V^{\lambda,\mu}$.

Case 2. If only one of V_1 or V_2 is self-associate, then there are two distinct irreducible submodules of $V_0 \otimes V_1 \otimes V_2$. Now our notation $V^{\lambda,\mu}$ is somewhat sloppy in this case as there are two choices for the submodules. In situations where we need to emphasize the existence of these two choices, we will write $V_{\pm}^{\lambda,\mu}$. We have $V_0 \otimes V_1 \otimes V_2 = V_{+}^{\lambda,\mu} \oplus V_{-}^{\lambda,\mu}$.

Case 3. If both V_1 and V_2 are self-associate, then $V_0 \otimes V_1 \otimes V_2$ is a direct sum of two copies of one irreducible module, so in this case there is only one choice for $V^{\lambda,\mu}$ (up to isomorphism). We have $V_0 \otimes V_1 \otimes V_2 = V^{\lambda,\mu} \oplus V^{\lambda,\mu}$.

As a summary, we have the following lemma.

Lemma 3.5.1 *With above notations, we have*

Case 1. If $\lambda, \mu \in DP^{-}$, *then* $V_0 \otimes V_1 \otimes V_2 = V^{\lambda,\mu}$.

Case 2. If $\lambda \in DP^{+}, \mu \in DP^{-}$ *or* $\lambda \in DP^{-}, \mu \in DP^{+}$, *then* $V_0 \otimes V_1 \otimes V_2 = V_{+}^{\lambda,\mu} \oplus V_{-}^{\lambda,\mu}$, *and* $V^{\lambda,\mu} = V_{+}^{\lambda,\mu}$ *or* $V_{-}^{\lambda,\mu}$.

Case 3. If $\lambda, \mu \in DP^{+}$, *then* $V_0 \otimes V_1 \otimes V_2 = V^{\lambda,\mu} \oplus V^{\lambda,\mu}$.

In the following, we define $\Theta^{\lambda,\mu}$ to be the representation afforded by $V^{\lambda,\mu}$ and $\theta^{\lambda,\mu}$ to be the corresponding character.

We define

$$\Phi^{\lambda,\mu} := \Theta^{\lambda,\mu} \uparrow W_n[\alpha].$$

In Case 2, i.e., $\lambda \in DP^{+}, \mu \in DP^{-}$ or $\lambda \in DP^{-}, \mu \in DP^{+}$, there is actually a pair of ε-associates $\Phi_{\pm}^{\lambda,\mu}$, namely

$$\Phi_{+}^{\lambda,\mu} = \Theta_{+}^{\lambda,\mu} \uparrow W_n[\alpha],$$

$$\Phi_{-}^{\lambda,\mu} = \Theta_{-}^{\lambda,\mu} \uparrow W_n[\alpha].$$

Theorem 3.5.2 *The irreducible spin representations of* $W_n[1,1,-1,1,]$ *are* $\Phi^{\lambda,\mu}$ *(for* $\varepsilon(\lambda,\mu)=1$*, i.e., Case 1 and Case 3) and* $\Phi_{\pm}^{\lambda,\mu}$ *(for* $\varepsilon(\lambda,\mu)=-1$*, i.e., Case 2), where* $\lambda,\mu \in DP$ *and* $|\lambda|+|\mu|=n$*.*

Proof: This is just Theorem 7.1 of Stembridge [23]. ■

As usual, we use $\varphi^{\lambda,\mu}$ to denote the character of $\Phi^{\lambda,\mu}$. In Case 2 we have $\varphi_{\pm}^{\lambda,\mu}$. The next proposition lists some properties about $\varphi^{\lambda,\mu}$ which will be used later when we study the Schur index of $\varphi^{\lambda,\mu}$. But we give the following definition first.

Definition 3.5.3 *For any partition* λ *of* n*, it is sometimes convenient to write* $\lambda = (1^{m_1}2^{m_2}\cdots)$ *to indicate that* m_j *of the parts of* λ *are size* j*. In these terms, we set*

$$Z_\lambda = (m_1!1^{m_1})(m_2!2^{m_2})\cdots$$

then Z_λ *is the centralizer of any permutation of type* λ*.*

Proposition 3.5.4 *The following provides some information about* $\varphi^{\lambda,\mu}(\alpha,\beta)$*. Here* α *and* β *denote the partitions such that* $|\alpha|+|\beta|=n$*.*

1. Split conjugacy classes:

For the case [1,1,-1,1,1], conjugacy class splits only for the following cases:

$$\begin{aligned}
\varepsilon &= +1,\ \delta=+1\ (OP,OP)\\
\varepsilon &= +1,\ \delta=-1\ (OP,OP)\\
\varepsilon &= -1,\ \delta=+1\ (DP,DP)\\
\varepsilon &= -1,\ \delta=-1\ (DP,DP).
\end{aligned}$$

2. Define $c_{\lambda,\mu}=2$ *for* $\lambda,\mu\in DP^-$*, and* $c_{\lambda,\mu}=1$*, otherwise.*

(a) If $\alpha\ \ \beta \in OP$*, then*

$$\varphi^{\lambda,\mu}(\alpha,\beta) = c_{\lambda,\mu}\sum_{I,J}(-1)^{|J^c|}\varphi^{\lambda}(\alpha_I\cup\beta_J)\varphi^{\mu}(\alpha_I^c\cup\beta_J^c),$$

where the summation is over subsets I *and* J *such that* $|\alpha_I\cup\beta_J|=|\lambda|$*.*

(b) If $\alpha, \beta \in DP$, and $\varepsilon(\alpha, \beta) = -1$, then $\varphi_{\pm}^{\lambda,\mu}(\alpha, \beta) = 0$ unless $\lambda \cup \mu = \alpha \cup \beta$. In that case, we have

$$\varphi_{\pm}^{\lambda,\mu}(\alpha, \beta) = \pm 2^{l(\lambda \cap \mu)} \varepsilon_{\alpha,\beta}^{\lambda,\mu} i^{(n-l(\lambda)-l(\mu)-1)/2} \sqrt{\frac{1}{2} Z_\lambda Z_\mu},$$

where $\varepsilon^{\lambda,\mu}$ is as defined in Stembridge [23].

3. Define Θ to be the representation afforded by $V_0 \otimes (V_1 \otimes V_2)$ and θ be the associated character. Then

$$\theta \uparrow W_n[\alpha] = \begin{cases} \varphi^{\lambda,\mu}, & \lambda \ \ \mu \in DP^-. \\ \varphi_+^{\lambda,\mu} + \varphi_-^{\lambda,\mu}, & \lambda \in DP^+, \ \mu \in DP^- \text{ or } \ \lambda \in DP^-, \ \mu \in DP^+. \\ 2\varphi^{\lambda,\mu}, & \lambda, \ \mu \in DP^+. \end{cases}$$

4. $\varphi^{\lambda,\mu}$ has the following relations related to the linear characters.

$$\begin{aligned} \varepsilon\varphi^{\lambda,\mu} &= \varphi^{\lambda,\mu} \text{ iff } \lambda, \ \mu \in DP^- \text{ or } \lambda, \ \mu \in DP^+. \\ \delta\varphi^{\lambda,\mu} &= \varphi^{\lambda,\mu} \text{ iff } \lambda = \mu. \end{aligned}$$

In the case $\lambda = \mu$, $\varphi^{\lambda,\mu}$ is also self-associate with respect to ε.

Proof: Conclusion 1 is part of Theorem 2.3.1; Conclusion 2 follows from Theorem 7.6 of Stembridge [23]; Conclusion 3 follows from the definition of $\varphi^{\lambda,\mu}$; and Conclusion 4 also follows from the definition of $\varphi^{\lambda,\mu}$, or readers may refer to page 433 of Stembridge [23]. ■

Corollary 3.5.5 *Some properties about $\varphi^{\lambda,\mu}$:*

(1) $\mathbf{Q}\left(\varphi^{\lambda,\mu}\right) = \mathbf{Q}$.

$$\mathbf{Q}\left(\varphi_{\pm}^{\lambda,\mu}\right) = \mathbf{Q}\left(i^{(n-l(\lambda)-l(\mu)-1)/2}\sqrt{\tfrac{1}{2} Z_\lambda Z_\mu}\right).$$

(2) $\varepsilon\varphi_+^{\lambda,\mu} = \varphi_-^{\lambda,\mu}$, $\varepsilon\varphi_-^{\lambda,\mu} = \varphi_+^{\lambda,\mu}$. I.e., $\varphi_+^{\lambda,\mu}$ and $\varphi_-^{\lambda,\mu}$ form an ε-associate pair.

Corollary 3.5.6 *$\varphi^{\lambda,\mu}$ has following relations with respect to linear characters:*

$$L(\varphi^{\lambda,\mu}) = \begin{cases} \{1\} & \lambda \in DP^- \textit{ and } \mu \in DP^+ \text{or } \lambda \in DP^+ \textit{ and } \mu \in DP^-. \\ \{1, \varepsilon\} & \text{if } \lambda, \mu \in DP^- \text{ or } \lambda, \mu \in DP^+, \text{ but } \lambda \neq \mu. \\ \{1, \varepsilon, \delta, \varepsilon\delta\} & \text{if } \lambda = \mu. \end{cases}$$

Corollary 3.5.7 *If $\lambda, \mu \in DP^+$ or $\lambda, \mu \in DP^-$, then $r\varphi^{\lambda,\mu}$ can be realized over $\mathbf{Q}$, otherwise, $r(\varphi_+^{\lambda,\mu} + \varphi_-^{\lambda,\mu})$ can be realized over $\mathbf{Q}$, for some positive integer r.*

Proof: This follows from Corollary 3.2.5 and the definition of $\varphi^{\lambda,\mu}$. ■

Corollary 3.5.8 *If* $\lambda, \mu \in DP^-$ or $\lambda, \mu \in DP^+$, but $\lambda \neq \mu$, *let H be the kernel of ε, then the graded algebra $A = End_{\mathbf{Q}H}(M)$ is a* **CSGA** *of odd type, where M is a $\varphi^{\lambda,\mu}$-quasihomogeneous $\mathbf{Q}W_n[\alpha]$-module.*

If $\lambda = \mu$, let H_1 be the kernel of ε, H_2 be the kernel of δ, then the two graded algebras $A := End_{\mathbf{Q}H_1}(M_1)$ and $B := End_{\mathbf{Q}H_2}(M_2)$ are **CSGA** *of odd type, where M_i is a $\varphi^{\lambda,\mu}$-quasihomogeneous $\mathbf{Q}W_n[\alpha]$-module, $i = 1, 2$.*

Proof: If $\lambda, \mu \in DP^-$ or $\lambda, \mu \in DP^+$, but $\lambda \neq \mu$, then $\mathbf{Q}(\varphi^{\lambda,\mu}) = \mathbf{Q}$. By Theorem 2.6.8 we conclude that the graded algebra $A = End_{\mathbf{Q}H}(M)$ is a **CSGA**. Finally, by Proposition 2.6.10 A is of odd type.

Similarly, if $\lambda = \mu$, then $\mathbf{Q}(\varphi^{\lambda,\mu}|_{H_i}) = \mathbf{Q}$. By Theorem 2.6.8 we conclude that both A and B are **CSGA**. Finally, by Proposition 2.6.10 A and B are of odd type.

■

In addition to the above information on $\varphi^{\lambda,\mu}$, we need some information about the associators for each of four linear representations $1, \varepsilon, \delta, \varepsilon\delta$. In what follows we will carry over the notations from Stembridge [23].

Theorem 3.5.9 *Assume $\lambda = \mu$ and M is a $\varphi^{\lambda,\mu}$-quasihomogeneous $\mathbf{Q}W_n[\alpha]$-module. Let H_1 be the kernel of ε and H_2 be the kernel of δ. Then both graded algebras $A := End_{\mathbf{Q}H_1}(M)$ and $B := End_{\mathbf{Q}H_2}(M)$ are* **CSGA** *of odd type. Let*

$$\begin{aligned} A &\simeq [D_1, 1, d_1], A_0 = A_0(M) \\ Z(A) &\simeq \mathbf{Q} + i\mathbf{Q}, i^2 \in \mathbf{Q}^\times. \end{aligned}$$

Note that i is an ε-associator.

$$\begin{aligned} B &\simeq [D_2, 1, d_2], B_0 = A_0(M) \\ Z(B) &\simeq \mathbf{Q} + j\mathbf{Q}, j^2 \in \mathbf{Q}^\times. \end{aligned}$$

Note that j is a δ-associator.

Then

1. *i and j anti-commute iff $l(\lambda)$ is odd iff $\lambda \in DP^+$ and $n \equiv 2$ (4) or $\lambda \in DP^-$ and $n \equiv 0$ (4).*

2. *i and j commute iff $l(\lambda)$ is even iff $\lambda \in DP^+$ and $n \equiv 0$ (4) or $\lambda \in DP^-$ and $n \equiv 2$ (4).*

Proof: Stembridge, in [23], already found associators for each of the linear representations. His work is done over $\mathbf{C}$. Since we are considering the modules over $\mathbf{Q}$, we need to modify Stembridge's definitions a little bit to make them suitable to our case.

We modify Stembridge's definitions about $S^{\lambda,\mu}$, $E_{\pm}^{\lambda,\mu}$ and $T_{\pm}^{\lambda}$ by the following (see p434 - p445 of Stembridge [23])

$$S^{\lambda,\mu}(\omega_0 \otimes v_0 \otimes v_1 \otimes v_2) = \begin{cases} \varepsilon(\omega_0)\omega_0 \otimes y_0 x_0 v_0 \otimes v_1 \otimes v_2 & \text{if } \lambda, \mu \in DP^- \\ \varepsilon(\omega_0)\omega_0 \otimes v_0 \otimes S_1 v_1 \otimes S_2 v_2 & \text{if } \lambda, \mu \in DP^+, \end{cases}$$

$$E_{\pm}^{\lambda,\mu} = 1 \otimes x_0 \otimes 1 \otimes S_2 \pm 1 \otimes y_0 \otimes S_1 \otimes 1,$$

$$T_{\pm}^{\lambda}(\omega_0 \otimes v_0 \otimes v_1 \otimes v_2) = \omega_0 u \otimes (x_0 \pm y_0) v_0 \otimes v_2 \otimes v_1,$$

and

$$T^{\lambda} = \begin{cases} T_{+}^{\lambda} & \text{if } n = 0 \bmod 4 \text{ and } \lambda \in DP^- \\ E_{+}^{\lambda,\lambda} T_{+}^{\lambda} & \text{if } n = 0 \bmod 4 \text{ and } \lambda \in DP^+ \\ T_{-}^{\lambda} & \text{if } n = 2 \bmod 4 \text{ and } \lambda \in DP^- \\ E_{-}^{\lambda,\lambda} T_{-}^{\lambda} & \text{if } n = 0 \bmod 4 \text{ and } \lambda \in DP^-. \end{cases}$$

where $u \in S_n$ denotes the involution $(1, k+1)(2, k+2) \cdots (k, 2k)$.

Then following the same proofs of Theorem 8.1 and Theorem 8.3 of Stembridge [23], we conclude that $S^{\lambda,\mu}$ is an ε-associator, $[S^{\lambda,\mu}, A_0(M)] = 1$ and T^{λ} is a δ-associator. Furthermore

1. $S^{\lambda,\lambda}$ and T^{λ} anti-commute iff $l(\lambda)$ is odd iff $\lambda \in DP^+$ and $n \equiv 2$ (4) or $\lambda \in DP^-$ and $n \equiv 0$ (4).

2. $S^{\lambda,\lambda}$ and T^{λ} commute iff $l(\lambda)$ is even iff $\lambda \in DP^{+}$ and $n \equiv 0$ (4) or $\lambda \in DP^{-}$ and $n \equiv 2$ (4).

Finally, it is easy to show $A_1 = iA_0$, similarly $B_1 = jB_0$. This implies that

$$S^{\lambda,\mu} = fi, \text{for some } f \in A_0(M)$$
$$T^{\lambda} = f'j, \text{for some } f' \in A_0(M).$$

Since $[i, A_0(M)] = 1 = [S^{\lambda,\mu}, A_0(M)]$ we get $f = S^{\lambda,\mu} i^{-1} \in A_0 \cap Z(A_0) = \mathbf{Q}$. This fact further forces

$$[S^{\lambda,\mu}, T^{\lambda}] = [f, f'][i, j] = [i, j].$$

Then our desired results follow and the proof of the theorem is finished. ■

Let $\alpha = [1, 1, -1, 1, 1]$ and denote

$$\begin{aligned} G_1 &= W_k[\alpha],\ H_1 = ker(\varepsilon), \\ G_2 &= W_{n-k}[\alpha],\ H_2 = ker(\varepsilon), \\ G_3 &= W_{k,n-k}[\alpha],\ H_3 = ker(\varepsilon), \end{aligned}$$

then we have four group pairs $(G_1, H_1), (G_2, H_2), (G_3, H_3)$ and (G, H). Here ε denotes the corresponding "sign" character of each group pair. Even though we use the same notation, it will be easily understood from the context. Let's carry over the notations from the definition of $\varphi^{\lambda,\mu}$. The next theorem gives us some relations between these group pairs in terms of the "$\vee$" operation (recall the definition of "$\vee$" in Section 2.7).

Theorem 3.5.10 *With the above notations, we have $G_3 \simeq G_1 \vee G_2$; i.e., $W_{k,n-k}[\alpha] \simeq W_k[\alpha] \vee W_{n-k}[\alpha]$, and $V_0 \otimes V_1 \otimes V_2 \simeq V_1 \vee V_2$, and the character χ afforded by $V_0 \otimes V_1 \otimes V_2$ is $\chi_1 \vee \chi_2$, where χ_i denotes the character afforded by V_i, $i = 1, 2$, i.e., $\chi = \chi_1 \vee \chi_2$.*

Proof: Let

$$W_k[\alpha] = \langle \sigma_1, \ldots, \sigma_{k-1}, \tau_1, \ldots, \tau_k, -1 \rangle,$$

$$\begin{aligned} W_{n-k}[\alpha] &= \langle \sigma_{k+1}, \ldots, \sigma_{n-1}, \tau_{k+1}, \ldots, \tau_n, -1 \rangle, \text{ and} \\ W_{k,n-k}[\alpha] &= \langle \sigma_i, \tau_j, -1 | i \neq k, j = 1, \cdots, n \rangle. \end{aligned}$$

then

$$\begin{aligned} W_k[\alpha] &\trianglelefteq W_{k,n-k}[\alpha], \quad W_{n-k}[\alpha] \trianglelefteq W_{k,n-k}[\alpha], \\ W_k[\alpha]W_{n-k}[\alpha] &= W_{k,n-k}[\alpha] \text{ and } W_k[\alpha] \cap W_{n-k}[\alpha] = \{1, -1\}. \end{aligned}$$

The only thing remains is to check that

$$g_1 g_2 = g_2 g_1 \text{ unless both } g_1 \notin H_1 \text{ and } g_2 \notin H_2$$

and that

$$g_1 g_2 = (-1) g_2 g_1 \text{ if } g_1 \notin H_1 \text{ and } g_2 \notin H_2.$$

Let $g_1 \in G_1, g_2 \in G_2$. Then g_1 is a product of some σ_i's and τ_j's where $i = 1, \cdots, k-1, j = 1, \cdots, k$. Similarly, g_2 is a product of some σ_i's and τ_j's where $i = k+1, \cdots, n-1, j = k+1, \cdots, n$. Assume g_1 has m σ's in its expression and g_2 has l σ's in its expression. Since $\alpha = [1, 1, -1, 1, 1]$, by Proposition 2.1.3 we have

$$g_1 g_2 = (-1)^{ml} g_2 g_1,$$

then our desired result follows.

Therefore, by the definition of $''\vee''$ in Section 2.7 (or refers to Turull [25]), we have

$$G_3 \simeq G_1 \vee G_2, \ \textit{i.e.}, W_{k,n-k}[\alpha] \simeq W_k[\alpha] \vee W_{n-k}[\alpha].$$

Also we have

$$\begin{aligned} H_3 &\simeq \{g_1 g_2 | \text{either both } \ g_1 \in H_1, g_2 \in H_2 \\ &\qquad \text{or both } \ g_1 \in G_1 \setminus H_1, g_2 \in G_2 \setminus H_2\} \\ &= \text{The distinguished subgroup of } G_1 \vee G_2 \end{aligned}$$

(recall the definition of distinguished subgroup in Section 2.7).

$V_0 \otimes V_1 \otimes V_2$ affords character

$$\begin{aligned}\chi(g_1g_2) &= \begin{cases} 0 & \text{if either } g_1 \notin H_1 \text{ or } g_2 \notin H_2 \\ 2\chi_1(g_1)\chi_2(g_2) & \text{if both } g_1 \in H_1 \text{ and } g_2 \in H_2 \end{cases} \\ &= \chi_1 \vee \chi_2.\end{aligned}$$

Therefore

$$V_0 \otimes V_1 \otimes V_2 \simeq V_1 \vee V_2.$$

The proof of the theorem is now complete. ■

Corollary 3.5.11 *We carry over the notations from the definition of $\varphi^{\lambda,\mu}$. As before, let $\theta^{\lambda,\mu}$ be the character associated with an irreducible constituent of $V_0 \otimes V_1 \otimes V_2$, then*

$$\begin{aligned}[[\theta^{\lambda,\mu}]] &= [[\varphi^{\lambda}]] \vee [[\varphi^{\mu}]] \\ &= [[\theta^{\lambda}]] \vee [[\theta^{\mu}]],\end{aligned}$$

where θ^{λ} and θ^{μ} are the irreducible characters associated with $\tilde{S}_{\lambda}$ and $\tilde{S}_{\mu}$ respectively, see p95-96 of Turull [25] or p109 of Stembridge [22].

Proof: By Corollary 2.7.3, we have

$$A(V_1 \vee V_2) \simeq A(V_1)\hat{\otimes}A(V_2)\hat{\otimes}\mathbf{Q}[i]$$

or

$$A(V_1 \vee V_2) \simeq A(V_1) \vee A(V_2).$$

Since $A(V_1), A(V_2)$ are **CSGA**, so is $A(V_1) \vee A(V_2)$, hence $A(V_1 \vee V_2)$ is a **CSGA**. By Theorem 2.6.8, the character afforded by $V_1 \vee V_2$ is $\theta^{\lambda,\mu}$-quasihomogenous. Therefore $[[\theta^{\lambda,\mu}]] = A(V_1 \vee V_2)$. Similarly, we have $[[\varphi^{\lambda,\emptyset}]] = A(V_1)$ and $[[\delta\varphi^{\mu,\emptyset}]] = A(V_2)$. It then follows

$$[[\theta^{\lambda,\mu}]] = [[\varphi^{\lambda,\emptyset}]] \vee [[\delta\varphi^{\mu,\emptyset}]].$$

Note that $[[\delta\varphi^{\mu,\emptyset}]] = [[\varphi^{\mu}]]$ because δ is a linear character, so we get

$$[[\theta^{\lambda,\mu}]] = [[\varphi^{\lambda}]] \vee [[\varphi^{\mu}]].$$

Finally, by the proof of Theorem 6.2 of Turull [25], we have $[[\varphi^{\lambda}]] = [[\theta^{\lambda}]]$ and $[[\varphi^{\mu}]] = [[\theta^{\mu}]]$, then our desired result follows. ■

Proposition 3.5.12 . *Let $\lambda \in DP_k, \mu \in DP_{n-k}$. Assume $k \geq 2$ and $n-k \geq 2$. Then every irreducible character of $\theta^{\lambda,\mu}|_{H_3}$ is in $\varphi^{\lambda,\mu}|_{H_3}$ with multiplicity one. Furthermore, if $\lambda,\ \mu \in DP^-$ or $\lambda,\ \mu \in DP^+$, then both $\varphi^{\lambda,\mu}|_H$ and $\theta^{\lambda,\mu}|_{H_3}$ are the sum of two irreducible characters; otherwise, both $\varphi^{\lambda,\mu}|_H$ and $\theta^{\lambda,\mu}|_{H_3}$ are irreducible.*

Proof: $k \geq 2$ and $n-k \geq 2$ make it true that both (G_1, H_1) and (G_2, H_2) are group pairs.

Note that $\varphi^{\lambda,\mu}$ is self-associate with respect to ε iff $\theta^{\lambda,\mu}$ is self-associate with respect to ε. This can be verified by considering the three cases for $\lambda,\ \mu : \lambda, \mu \in DP^-; \lambda,\ \mu \in DP^+$; and otherwise. Then the last two sentences follow.

Since

$$\theta^{\lambda,\mu}|^G = \varphi^{\lambda,\mu},\ [\theta^{\lambda,\mu}, \varphi^{\lambda,\mu}|_{G_3}] = [\theta^{\lambda,\mu}|^G, \varphi^{\lambda,\mu}] = [\varphi^{\lambda,\mu}, \varphi^{\lambda,\mu}] = 1,$$

we have

$$\varphi^{\lambda,\mu}|_{G_3} = \theta^{\lambda,\mu} + \psi + \ldots (\theta^{\lambda,\mu} \neq \psi).$$

Therefore, $\theta^{\lambda,\mu}|_{H_3}$ is in $\varphi^{\lambda,\mu}|_{H_3}$.

Let θ_1 be any irreducible constitute of $\theta^{\lambda,\mu}|_{H_3}$, we need to show that θ_1 is in $\varphi^{\lambda,\mu}|_{H_3}$ with multiplicity one.

Case 1. Both $\theta^{\lambda,\mu}|_{H_3}$ and $\varphi^{\lambda,\mu}|_H$ are irreducible. In this case we have $\theta_1 = \theta^{\lambda,\mu}|_{H_3}$ and

$$\theta^{\lambda,\mu} = \theta_+^{\lambda,\mu},\ \varphi^{\lambda,\mu} = \varphi_+^{\lambda,\mu} = \theta_+^{\lambda,\mu}|^G.$$

If $[\theta_1, \psi|_{H_3}] \neq 0$, then $[\theta_1^{G_3}, \psi] \neq 0$, by Gallagher's Theorem 6.7 of Isaacs [9], we get

$$\theta_1^{G_3} = \theta_+^{\lambda,\mu} + \varepsilon\theta_+^{\lambda,\mu} = \theta_+^{\lambda,\mu} + \theta_-^{\lambda,\mu},$$

hence $[\theta_+^{\lambda,\mu} + \theta_-^{\lambda,\mu}, \psi] \neq 0$. This implies $[\theta_-^{\lambda,\mu}, \psi] \neq 0$ and therefore $\psi = \theta_-^{\lambda,\mu}$. It follows $[\varphi_+^{\lambda,\mu}|_{G3}, \theta_-^{\lambda,\mu}] \neq 0$. On the other hand, $[\varphi_+^{\lambda,\mu}|_{G_3}, \theta_-^{\lambda,\mu}] = [\varphi_+^{\lambda,\mu}, \theta_-^{\lambda,\mu}|^G] = [\varphi_+^{\lambda,\mu}, \varphi_-^{\lambda,\mu}] = 0$, a contradiction. Therefore, $[\theta_1, \psi|_{H_3}] = 0$. So in this case, θ_1 is in $\varphi^{\lambda,\mu}|_{H_3}$ with multiplicity one.

Case 2. Both $\theta^{\lambda,\mu}|_{H_3}$ and $\varphi^{\lambda,\mu}|_H$ are the sum of two distinct irreducible characters. Let $\theta^{\lambda,\mu}|_{H_3} = \theta_1 + \theta_2$, then $I_{G_3}(\theta_1) = I_{G_3}(\theta_2) = H_3$, here $I_{G_3}(\theta_1)$ and $I_{G_3}(\theta_2)$ denote the inertia groups of θ_1 and θ_2. By 6.11 of Isaacs [9], $\theta_1^{G_3} = \theta_2^{G_3} = \theta^{\lambda,\mu}$. If $[\theta_1, \psi|_{H_3}] \neq 0$, then $[\theta_1^{G_3}, \psi] = [\theta^{\lambda,\mu}, \psi] \neq 0$, a contradiction. Therefore, θ_1 (and θ_2) is in $\varphi^{\lambda,\mu}|_{H_3}$ with multiplicity one.

This completes the proof of the proposition. ■

Theorem 3.5.13 *Let $\lambda \in DP_k$, $\mu \in DP_{n-k}$. Assume $k \geq 2, n-k \geq 2$. Then*

$$[[\varphi^{\lambda,\mu}]] = [[\theta^{\lambda,\mu}]].$$

Proof: Let $M = (V_0 \otimes V_1 \otimes V_2) \uparrow G$, then M is a $\varphi^{\lambda,\mu}$-quasihomogeneous $\mathbf{Q}G$-module and $[[\varphi^{\lambda,\mu}]]$ is the class in $BW(\mathbf{Q})$ of $A(M) := End_{\mathbf{Q}H}(M)$.

Let N be the sum of all irreducible $\mathbf{Q}G_3$-submodules of M which are $\theta^{\lambda,\mu}$-quasihomogeneous. Then $[[\theta^{\lambda,\mu}]]$ is the class in $BW(\mathbf{Q})$ of $A(N) := End_{\mathbf{Q}H_3}(N)$.

If $f \in A(M)$, it follows that $f(N) \subseteq N$. Hence, we define the restriction map

$$\begin{aligned} Res : A(M) &\longrightarrow A(N) \\ Res(f) : N &\longrightarrow N \\ Res(f)(n) &= f(n). \end{aligned}$$

Since f commutes with H, it also commutes with H_3, so that Res is well-defined. Furthermore, routine arguments show that, in fact, Res is a graded algebra homomorphism. By Theorem 2.6.8, $A(M)$ is a central simple graded $\mathbf{Q}$-algebra and so is $A(N)$. It follows that Res is injective.

Let χ be the character afforded by M. Then $dim_{\mathbf{Q}} A(M) = [\chi|_H, \chi|_H]_H$.

Suppose $\lambda \in DP^+$ and $\mu \in DP^-$ or $\lambda \in DP^-$ and $\mu \in DP^+$, then $\varphi^{\lambda,\mu} = \varphi_+^{\lambda,\mu}, \theta^{\lambda,\mu} = \theta_+^{\lambda,\mu}$ and both $\varphi^{\lambda,\mu}|_{H_3}$ and $\theta^{\lambda,\mu}|_{H_3}$ are irreducible, and $\theta^{\lambda,\mu}|_{H_3}$ is in $\varphi^{\lambda,\mu}|_{H_3}$ with multiplicity one. Hence, $\chi|_H = r\varphi^{\lambda,\mu}|_H$, for some r, and $dim_{\mathbf{Q}} A(M) = r^2$. Furthermore, the character afforded by N as an H_3-module is $r\theta^{\lambda,\mu}|_{H_3}$ ($H_3 = H \cap G_3$). Hence $dim_{\mathbf{Q}} A(N) = r^2$, and Res is an isomorphism.

Suppose now $\lambda, \mu \in DP^+$ or $\lambda, \mu \in DP^-$, then $\varphi^{\lambda,\mu}|_H$ and $\theta^{\lambda,\mu}|_{H_3}$ both are sum of two distinct irreducible characters. Hence $\chi|_H = r\varphi^{\lambda,\mu}|_H$, for some r, and the character afforded by N as H_3-module is in $r\theta^{\lambda,\mu}|_{H_3}$. Hence $dim A(M) = 2r$ and $dim A(N) = 2r$. We conclude that Res is a graded algebra isomorphism in all cases. It follows that $[[\varphi^{\lambda,\mu}]] = [[\theta^{\lambda,\mu}]]$. ■

Theorem 3.5.14 . *Let* $\lambda \in DP_k$, $\lambda = (\lambda_1 > \lambda_2 > \ldots > \lambda_l)$, $\mu \in DP_{n-k}$, $\mu = (\mu_1 > \mu_2 > \ldots > \mu_m)$. *Then*

$$[[\varphi^{\lambda,\mu}]] = [\lambda_1] \vee \ldots \vee [\lambda_l] \vee [\mu_1] \vee \ldots \vee [\mu_m],$$

where

$$[n] = \left[(-1,-1)^{\binom{n+1}{4}} \left(-2(-1)^{\frac{n+1}{2}}, (-1)^{\frac{n-1}{2}} n\right), 1, n(-1)^{\frac{n-1}{2}}\right] \text{ if } n \text{ is odd.}$$

and

$$[n] = \left[(-1,-1)^{\binom{n+1}{4}} \left((-1)^{\frac{n}{2}}, n\right), 0, -2(-1)^{\frac{n}{2}} n\right] \text{ if } n \text{ is even.}$$

Proof: If $k \geq 2, n-k \geq 2$, by the previous theorem, we get

$$[[\varphi^{\lambda,\mu}]] = [[\theta^{\lambda,\mu}]] = [[\theta^{\lambda}]] \vee [[\theta^{\mu}]].$$

Then by Theorem 6.2 of Turull [25], we have

$$[[\theta^{\lambda}]] = [\lambda_1] \vee \ldots \vee [\lambda_l],$$

and

$$[[\theta^{\mu}]] = [\mu_1] \vee \ldots \vee [\mu_m].$$

Notice that φ^{λ} corresponds to the case $\varepsilon = -1$ in [20], then result follows.

If $k = 1$ or $n - k = 1$, from the definition of $\varphi^{\lambda,\mu}$, it is easy to see, for example $k = 1$ (i.e., $|\lambda| = 1$)

$$[[\varphi^{\lambda,\mu}]] = [[\varphi^{\emptyset,\mu}]] = [[\delta\varphi^{\mu,\emptyset}]] = [[\varphi^{\mu,\emptyset}]] = [[\varphi^{\mu}]]$$

because [1] is described by the triple [1,1,1], which is the identity under "$\vee$" operation. So Theorem holds also in this case. Similarly, when $n - k = 1$, Theorem holds too. The proof is now complete. ∎

Corollary 3.5.15 *Assume* $[[\varphi^{\lambda,\mu}]] = [D, t, d]$ *is the triple associated with* $\varphi^{\lambda,\mu}$*, then* D *can be chosen to be a quaternion algebra.*

Proof: This follows from the above theorem and Theorem 2.5.4. ∎

Corollary 3.5.16 $\varphi^{\lambda,\mu}$ *has the following properties:*

(1) The Schur index of $\varphi^{\lambda,\mu}$ *is at most* 2 *over* $\mathbf{Q}$.

(2) If $\lambda, \mu \in DP^{+}$ *or* $\lambda, \mu \in DP^{-}$*, then* $2\varphi^{\lambda,\mu}$ *can be realized over* $\mathbf{Q}$.

(3) If $\lambda \in DP^{+}$ *and* $\mu \in DP^{-}$ *or* $\lambda \in DP^{-}$ *and* $\mu \in DP^{+}$*, then* $2(\varphi_{+}^{\lambda,\mu} + \varphi_{-}^{\lambda,\mu})$ *can be realized over* $\mathbf{Q}$.

Proof: This follows from the above Corollary and Proposition 2.6.1. ∎

Remarks : Given any partitions λ and μ such that $|\lambda| + |\mu| = n$, Turull's algorithm in [25] enables us to efficiently compute the triple that describes $[[\varphi^{\lambda,\mu}]]$ by the above theorem, then Proposition 2.6.1 can be applied to calculate the Schur index for each irreducible spin character $\varphi^{\lambda,\mu}$.

Definition 3.5.17 *We define* $\mathcal{D}_0^{\lambda,\mu}, \mathcal{D}_1^{\lambda,\mu}, \mathcal{D}_2^{\lambda,\mu}, d_0^{\lambda,\mu}, d_1^{\lambda,\mu}, d_2^{\lambda,\mu}$ *for* $\lambda, \mu \in OP$. *Let* $G = W_n[\alpha]$ *with* $\alpha = [1, 1, -1, 1, 1]$. *Let* H *be the kernel of* ε. *If* $\lambda \in DP^+$ *and* $\mu \in DP^-$ *or* $\lambda \in DP^-$ *and* $\mu \in DP^+$, *let* M *be a* $\mathbf{Q}G$*-module affording* $2(\varphi_+^{\lambda,\mu} + \varphi_-^{\lambda,\mu})$, *then* $A := End_{\mathbf{Q}H}(M)$ *is a* **CSGA** *of even type. We set*

$$\begin{aligned} A &\simeq [\mathcal{D}_0^{\lambda,\mu}, 0, d_0^{\lambda,\mu}], A_0 = A_G(M) \\ Z(A) &\simeq \mathbf{Q} + i\mathbf{Q},\ i^2 = d_0^{\lambda,\mu} \in \mathbf{Q}^\times. \end{aligned}$$

If $\lambda, \mu \in DP^+$ *or* $\lambda, \mu \in DP^-$, *let* M *be a* $\mathbf{Q}G$*-module affording* $2\varphi^{\lambda,\mu}$. *Let* H_1 *and* H_2 *be the kernels of* ε *and* δ, *respectively, then* $A := End_{\mathbf{Q}H_1}(M)$ *is a* **CSGA** *of odd type. We set*

$$\begin{aligned} A &\simeq [\mathcal{D}_1^{\lambda,\mu}, 1, d_1^{\lambda,\mu}], A_0 = A_G(M) \\ Z(A) &\simeq \mathbf{Q} + i\mathbf{Q},\ i^2 = d_1^{\lambda,\mu} \in \mathbf{Q}^\times. \end{aligned}$$

Furthermore, if $\lambda = \mu$, *then* $B := End_{\mathbf{Q}H_2}(M)$ *is also a* **CSGA** *of odd type. We set*

$$\begin{aligned} B &\simeq [\mathcal{D}_2^{\lambda,\mu}, 1, d_2^{\lambda,\mu}], B_0 = A_G(M) \\ Z(B) &\simeq \mathbf{Q} + j\mathbf{Q},\ j^2 = d_2^{\lambda,\mu} \in \mathbf{Q}^\times. \end{aligned}$$

$\mathcal{D}_0^{\lambda,\mu}, \mathcal{D}_1^{\lambda,\mu} = \mathcal{D}_2^{\lambda,\mu}, d_0^{\lambda,\mu}, d_1^{\lambda,\mu}$ *and* $d_2^{\lambda,\mu}$ *can be calculated by using Theorem 3.5.14.*

Remark : In the following study, we will need the above **CSGA**'s all the time, so whenever we use these notations throughout the dissertation, they keep the above meanings unless otherwise mentioned.

3.6 The Definition of $\Psi^{\varepsilon_1,\varepsilon_2}$

In this section, we consider the group $W_n[\varepsilon_1, \varepsilon_2, -1, -1, -1]$. We will define a special spin representation to be denoted by $\Psi^{\varepsilon_1,\varepsilon_2}$. This special spin representation combined with $X^{\lambda,\mu}$ later can be used to characterize all spin characters of the double cover $W_n[\varepsilon_1, \varepsilon_2, -1, -1, -1]$. To do this, we need some results on Clifford algebras.

This section can be divided into three parts. First we define Clifford algebra $b_n(\mathbf{K})$ over an arbitrary field $\mathbf{K}$ of characteristic zero and quote some well-known results about Clifford algebras. Then in the second part of this section, we consider the Clifford algebra $b_n(\mathbf{C})$ over the complex numbers $\mathbf{C}$. Using the Clifford algebra $b_n(\mathbf{C})$, we define a special spin representation $\Psi^{\varepsilon_1,\varepsilon_2}$ and discuss some properties of $\Psi^{\varepsilon_1,\varepsilon_2}$. Finally in the third part of this section, we consider the Clifford algebra $b_n(\mathbf{Q})$ over rational numbers $\mathbf{Q}$. We construct another special character $\Im$ of the group $W_n[\varepsilon_1,\varepsilon_2,-1,-1,-1]$, which has a close relation with the character $\psi^{\varepsilon_1,\varepsilon_2}$ of the representation $\Psi^{\varepsilon_1,\varepsilon_2}$. We finish this section by calculating the algebra associated with the character $\psi^{\varepsilon_1,\varepsilon_2}$, which is useful in Chapter Four.

Let $\mathbf{K}$ be a field of characteristic zero. We first study the Clifford algebra over $\mathbf{K}$.

Let V be an n-dimensional vector space over $\mathbf{K}$ endowed with a non-singular symmetric bilinear form B. We say that V is a quadratic space. Then one associates to V its Clifford algebra $b(V)$ (see Lam [12]). We also use the notation $b_n(\mathbf{K})$ for $b(V)$ whenever necessary to indicate that the Clifford algebra $b(V)$ depends on both n and the field $\mathbf{K}$. $b(V)$ is a central simple graded $\mathbf{K}$-algebra of dimension 2^n. If $\xi_1,\xi_2,\ldots,\xi_n$ is a basis for V, then $\xi_1,\xi_2,\ldots,\xi_n$ can be viewed as generators of $b(V)$ of degree 1, satisfying the relations :

$$\xi_i\xi_j+\xi_j\xi_i=2B(\xi_i,\xi_j) \text{ for all } 1\le i,j\le n.$$

A basis for $b(V)$ is given by $\xi_A=\xi_{a_1}\cdots\xi_{a_r}$ where $A=\{a_1,\ldots,a_r\}$ ranges over all subsets of $\{1,\ldots,n\}$. More precisely, the set of all ξ_A where $|A|$ is even is a basis of $b(V)_0$ and the set of all ξ_A where $|A|$ is odd is a basis of $b(V)_1$.

Proposition 3.6.1 *$b(V)$ is a central simple graded $\mathbf{K}$-algebra. The element of $BW(\mathbf{K})$ that it represents is described by the following triple,*

$$[c(V), dim_0(V), dim_{\pm}(V)],$$

where $dim_0(V)$ is the $dim(V)$ (modulo 2), and

$$dim_{\pm}(V) = (-1)^{\binom{n}{2}} d(V) \in (\mathbf{K})^{\times}/(\mathbf{K}^{\times})^2$$

where $d(V)$ is the determinant of the form B, and

$$c(V) = (-1, d(V))^{\binom{n-1}{2}} (-1,-1)^{\binom{n+1}{4}} s(V) \in Br(\mathbf{K}),$$

where $s(V)$ is the Hasse invariant of V. That is, if B can be represented by a diagonal matrix $diag(a_1, \ldots, a_n)$, then $s(V) = \prod_{i<j} (a_i, a_j)$.

Proof: This is just Proposition 5.1 of Turull [25]. ■

For our purpose, we are only interested in the case where $n \geq 2$ and $\xi_1, \xi_2, \ldots, \xi_n$ is an orthonormal basis, that is, $B(\xi_i, \xi_j) = 0$ if $i \neq j$ and $B(\xi_i, \xi_i) = 1$ for $i = 1, \ldots, n$. We use the notation $b_n(\mathbf{K}) = b(V)$ in this case. Sometimes we also simply use notation b_n instead of $b_n(\mathbf{K})$ if there is no confusion about the field $\mathbf{K}$ from the context.

Corollary 3.6.2 *b_n, as an element of $BW(\mathbf{K})$, is described by the following triple*

$$\left[(-1,-1)^{\binom{n+1}{4}}, dim_0(V), (-1)^{\binom{n}{2}}\right].$$

If n is even, $b(V)$ is a central simple $\mathbf{K}$-algebra (as an ungraded algebra), by the previous proposition, and we set C to be the ungraded algebra $b(V)$. If n is odd, $b(V)_0$ is a central simple $\mathbf{K}$-algebra, by the previous proposition, and we set $C = b(V)_0$. So in all cases, C is an (ungraded) central simple $\mathbf{K}$-algebra.

Lemma 3.6.3 *C is a representative of the element $(-1,-1)^{\binom{n+1}{4}}$ of $Br(\mathbf{K})$.*

Proof: This is just Lemma 5.2 of Turull [25]. ■

Lemma 3.6.4 *Denote $\xi = \prod \xi_i$, then*

1.

$$[\xi, \xi_i - \xi_{i+1}] = \begin{cases} 1 & \text{if } 2 \not| n \\ -1 & \text{if } 2 | n. \end{cases}$$

2.

$$[\xi, \xi_1] = \begin{cases} 1 & \text{if } 2 \not| n \\ -1 & \text{if } 2|n. \end{cases}$$

3.

$$\xi^2 = \begin{cases} (-1)^{\frac{n}{2}} & \text{if } 2|n \\ \\ (-1)^{\frac{n-1}{2}} & \text{if } 2 \not| n. \end{cases}$$

4.

$$(\xi_i - \xi_{i+1})\xi_j(\xi_i - \xi_{i+1}) = \begin{cases} -2\xi_{i+1} & \text{if } j = i \\ -2\xi_i & \text{if } j = i+1 \\ -2\xi_j & \text{otherwise.} \end{cases}$$

Proof: Checking is straightforward. ■

Next, we consider the Clifford algebra $b_n(\mathbf{C})$ or simply b_n over the complex numbers $\mathbf{C}$.

Stembridge, in [23], pages 446-447, considered the group $W_n[1, 1, -1, -1, -1]$ and defined an irreducible spin representation Ψ if n is even, $\Psi_\pm$ if n is odd by

$$\sigma_j \longmapsto \frac{1}{\sqrt{2}}(\xi_j - \xi_{j+1}), \tau \mapsto \xi_1.$$

Now we consider the group $W_n[\varepsilon_1, \varepsilon_2, -1, -1, -1]$. By modifying Stembridge's definition above a little bit, we are able to get irreducible spin representation $\Psi^{\varepsilon_1,\varepsilon_2}$ if n is even, $\Psi_\pm^{\varepsilon_1,\varepsilon_2}$ if n is odd by

$$\sigma_j \longmapsto \frac{i^{\gamma(\varepsilon_1)}}{\sqrt{2}}(\xi_j - \xi_{j+1}),$$

$$\tau \mapsto i^{\gamma(\varepsilon_2)}\xi_1,$$

where $\gamma(1) = 0$ and $\gamma(-1) = 1$.

The fact that this does generate an algebra homomorphism is an immediate consequence of the defining relations of $W_n[\varepsilon_1, \varepsilon_2, -1, -1, -1]$. We may thus obtain a representation of $W_n[\varepsilon_1, \varepsilon_2, -1, -1, -1]$ by composing $\mathbf{C}W_n[\varepsilon_1, \varepsilon_2, -1, -1, -1] \to b_n$ with any ordinary representation $b_n \to End(V)$.

If n is even, the algebra b_n is simple and thus isomorphic to the matrix algebra $M(2^{n/2})$. For odd n, we have $b_n \simeq M(2^{\frac{n-1}{2}}) \oplus M(2^{\frac{n-1}{2}})$ (see Lam [12]). It follows that b_n has one irreducible representation for even n, and two representations for odd n. Since $\mathbf{C}W_n[\varepsilon_1, \varepsilon_2, -1, -1, -1] \to b_n$ is surjective, the above construction yields one irreducible spin representation of $W_n[\gamma]$ for even n and two for odd n. We will denote these representations generally by $\Psi^{\varepsilon_1,\varepsilon_2}$, in any situation where we need to emphasize that there are two choices for odd n, we will write $\Psi^{\varepsilon_1,\varepsilon_2}_{\pm}$.

Let $\psi^{\varepsilon_1,\varepsilon_2}$ denote the character of $\Psi^{\varepsilon_1,\varepsilon_2}$. Then we have the following theorem.

Theorem 3.6.5 *For $\alpha \in OP$ and $\beta \in EP$, we have*

$$\psi^{\varepsilon_1,\varepsilon_2}(\alpha,\beta) = \begin{cases} i^{\gamma(\varepsilon_2)l(\beta)+\gamma(\varepsilon_1)[n-l(\alpha)-l(\beta)]}(-1)^{l(\beta)+\frac{n-l(\alpha)}{2}}2^{\frac{l(\alpha)+l(\beta)}{2}} & \text{if } 2|n \\ i^{\gamma(\varepsilon_2)l(\beta)+\gamma(\varepsilon_1)[n-l(\alpha)-l(\beta)]}(-1)^{\frac{n-l(\alpha)}{2}}2^{\frac{l(\alpha)+l(\beta)-1}{2}} & \text{if } 2 \not| n. \end{cases}$$

The only other non-zero values of $\psi^{\varepsilon_1,\varepsilon_2}(\alpha,\beta)$ occur when n is odd, $\alpha = \emptyset$, and β is arbitrary. In these cases, we have

$$\psi^{\varepsilon_1,\varepsilon_2}_{\pm}(\emptyset,\beta) = \pm(-1)^{\frac{(n-1)l(\beta)}{2}} i^{\gamma(\varepsilon_2)l(\beta)+\gamma(\varepsilon_1)[n-l(\beta)]+\frac{n-1}{2}} 2^{\frac{l(\beta)-1}{2}}.$$

Proof: We will follow the proof of Theorem 9.1 of Stembridge [22].

Take the elements $\xi_I = \xi_{i_1} \cdots \xi_{i_k}$ as a basis of b_n, where $I = \{i_1, \ldots, i_k\}$ ranges over the subsets of $\{1, \ldots, n\}$. For $\omega \in \mathbf{C}W_n[\gamma]$, let $\xi_I(\omega)$ denote the coefficient of ξ_I in the b_n-image of ω. By Proposition 3.1 of Stembridge [23] (a description of the irreducible character(s) of b_n), we have

$$\psi^{\varepsilon_1,\varepsilon_2}(\omega) = 2^{n/2}\xi_\emptyset(\omega) \qquad \text{if } n \text{ is even}$$

$$\psi^{\varepsilon_1,\varepsilon_2}_{\pm}(\omega) = \pm 2^{(n-1)/2}\xi_\emptyset(\omega) \pm (2i)^{(n-1)/2}\zeta(\omega) \quad \text{if } n \text{ is odd },$$

where $\zeta = \xi_1 \ldots \xi_n$ denotes the basis element corresponding to $I = \{1, \ldots, n\}$. $\zeta(\omega)$ denotes the coefficient of ζ in ω.

To evaluate $\xi_\emptyset(\omega)$, first consider the case in which ω is a positive, canonical k-cycle. If the S_n-image of ω is $(j+1, j+2, \ldots, j+k)$, then the b_n-image of ω is

$$2^{-(k-1)/2} i^{\gamma(\varepsilon_1)(k-1)}(\xi_{j+1} - \xi_{j+2}) \ldots (\xi_{j+k-1} - \xi_{j+k}), \tag{3.3}$$

so there is no constant term unless k is odd. In that case, only one of the 2^{k-1} terms that arise in the expansion of the above expression has a non-zero constant term. This single term is of the form $(-\xi_{j+2})(\xi_{j+2})(-\xi_{j+4})(\xi_{j+4})...$, so we conclude that

$$\xi_\emptyset(\omega) = i^{\gamma(\varepsilon_1)(k-1)}(-\frac{1}{2})^{(k-1)/2}.$$

To evaluate the constant term for a negative cycle, first note that the b_n-image of $\tau_j = \sigma_{j-1}^{-1}\cdots\sigma_1^{-1}\tau\sigma_1\cdots\sigma_{j-1}$ is $i^{r(\varepsilon_2)}(-1)^{j-1}\xi_j = (-1)^{j-1}i^{r(\varepsilon_2)}\xi_j$. This follows by induction on j and the fact that $(\xi_2 - \xi_1)\xi_1(\xi_2 - \xi_1) = -2\xi_2$. Therefore, the b_n-image of a negative, canonical k-cycle ω will be of the form

$$(-1)^{k+j-1}i^{r(\varepsilon_2)(k-1)}2^{-\frac{k-1}{2}}(\xi_{j+1} - \xi_{j+2})\cdots(\xi_{j+k-1} - \xi_{j+k}), \tag{3.4}$$

so there will be no constant term unless k is even. In that case, it is easy to see that

$$\xi_\emptyset(\omega) = (-1)^{[r(\varepsilon_1)+1](j-1)+\frac{k}{2}}i^{r(\varepsilon_2)(k-1)+r(\varepsilon_2)}(\frac{1}{2})^{\frac{k-1}{2}}.$$

For the general case, assume that ω is the canonical representative of some class (α, β), and let $\omega = \omega_1 \ldots \omega_l$ be its defining factorization as a product of canonical cycles. Note that $\xi_\emptyset(\omega) = \xi_\emptyset(\omega_1)\ldots\xi_\emptyset(\omega_l)$, we may therefore assume $\alpha \in OP$ and $\beta \in EP$ since the above analysis shows that $\xi_\emptyset(\omega)$ would otherwise be zero. Under these circumstance, the ith. negative cycle of ω_i (of length β_i) includes the element $\tau_{j+\beta_i}$ as part of its defining factorization, where $j = |\alpha| + \beta_1 + \ldots + \beta_{i-1}$. However, since $\alpha \in OP$ and $\beta \in EP$, it follows that $j = l(\alpha) = n$ (mod 2), and so we have

$$\begin{aligned}
\xi_\emptyset(\omega) &= \prod_{i=1}^{l(\alpha)} i^{\gamma(\varepsilon_1)(\alpha_i-1)}(-1/2)^{(\alpha_i-1)/2}\prod_{i=1}^{l(\beta)}(-1)^{n-1+\frac{\beta_i}{2}}i^{\gamma(\varepsilon_1)(\beta_i-1)+\gamma(\varepsilon_2)}(1/2)^{(\beta_i-1)/2} \\
&= i^{\gamma(\varepsilon_2)l(\beta)+\gamma(\varepsilon_1)[n-l(\alpha)-l(\beta)]}(-1)^{(n-1)l(\beta)+[n-l(\alpha)]/2}(1/2)^{[n-l(\alpha)-l(\beta)]/2}.
\end{aligned}$$

To evaluate $\zeta(\omega)$, observe that the b_n-image is a product of $n - l(\alpha)$ linear terms, where ζ is a product of n such terms. Hence, in order for $\zeta(\omega)$ to be non-zero,

we must have $\alpha = \emptyset$ (and thus $\beta \notin EP$, assuming n is odd). It is easy to see that the coefficient of $\xi_{j+1} \ldots \xi_{j+k}$ in (3.4) is

$$(-1)^{k+j-1} i^{\gamma(\varepsilon_2)+\gamma(\varepsilon_1)(k-1)} (1/2)^{(k-1)/2}.$$

So we have

$$\zeta(\omega) = \prod_{i=1}^{l(\beta)} (-1)^{\beta_1+\ldots\beta_i-1} i^{\gamma(\varepsilon_2)+\gamma(\varepsilon_1)(\beta_i-1)} (1/2)^{(\beta_i-1)/2}.$$

For odd n, we have $\beta_1 + \ldots + \beta_i - 1 = \beta_{i+1} + \beta_{i+2} \ldots \pmod 2$, so the above expression simplifies to

$$\begin{aligned} \zeta(\omega) &= (-1)^{\frac{(n-1)l(\beta)}{2}} i^{\gamma(\varepsilon_2)l(\beta)+\gamma(\varepsilon_1)[|\beta|-l(\beta)]} (1/2)^{\frac{n-l(\beta)}{2}} \\ &= (-1)^{\frac{(n-1)l(\beta)}{2}} i^{\gamma(\varepsilon_2)l(\beta)+\gamma(\varepsilon_1)[n-l(\beta)]} (1/2)^{\frac{n-l(\beta)}{2}}. \end{aligned}$$

(note $|\alpha| = 0, |\beta| = n$).

The claimed results now follow from the formulas $\xi_\emptyset(\omega)$ and $\zeta(\omega)$. ∎

Corollary 3.6.6 *The relations of $\psi^{\varepsilon_1,\varepsilon_2}$ with the linear characters can be described as follows.*

1. *If $2 \not| n$, $\psi^{\varepsilon_1,\varepsilon_2}$ is not self-associate w.r.t. neither ε, δ, nor $\varepsilon\delta$.*
2. *If $2|n$, $\psi^{\varepsilon_1,\varepsilon_2}$ is $\varepsilon\delta$-associate, hence $\varepsilon\psi^{\varepsilon_1,\varepsilon_2} = \delta\psi^{\varepsilon_1,\varepsilon_2}$.*

The results of the rest of this section are still true over the field $\mathbf{K}$ of characteristic zero. However, in our applications we are only interested in the field $\mathbf{Q}$. So let's fix the field $\mathbf{K} = \mathbf{Q}$ and consider the Clifford algebra $b_n(\mathbf{Q})$ or simply b_n over $\mathbf{Q}$.

Recall that the Clifford algebra b_n is a central simple graded algebra over $\mathbf{Q}$. The element of $BW(\mathbf{Q})$ that it represents is described by the following triple:

$$[c(V), dim_0(V), dim(V)] = \left[(-1,-1)^{\binom{n+1}{4}}, dim_0(V), (-1)^{\binom{n}{2}}\right].$$

Recall also that $C = b(V)$ if n is even and $C = b(V)_0$ if n is odd. In all cases, C is a central simple $\mathbf{Q}$-algebra and C is a representative of the element $(-1,-1)^{\binom{n+1}{4}}$ in $\mathrm{Br}(\mathbf{Q})$.

Proposition 3.6.7 *Set*

$$\begin{aligned}\gamma_i &= (\xi_i - \xi_{i+1})\xi_1 \ldots \xi_n \; for\; i = 1, 2, \ldots, n-1 \\ \gamma_0 &= \xi_2 \ldots \xi_n = \xi_1 \left(\prod_{i=1}^{n} \xi_i\right) = \xi_1 \xi.\end{aligned}$$

Then γ_i is an invertible element of C. Furthermore

(i) $\gamma_i^2 = -2(-1)^{\binom{n+1}{2}}$ *for* $i = 1, 2, \ldots, n-1$

(ii) $\gamma_0^2 = -(-1)^{\binom{n+1}{2}}$

(iii) $\gamma_i\gamma_{i+1}\gamma_i = \gamma_{i+1}\gamma_i\gamma_{i+1}$ for $i = 1, 2, \ldots, n-2$

(iv) $[\gamma_i, \gamma_j] = -1$ *if* $|i-j| \geq 2$

(v) $[\gamma_i, \gamma_0] = -1$ $(i > 1)$

(vi) $\gamma_1\gamma_0\gamma_1\gamma_0 = -\gamma_0\gamma_1\gamma_0\gamma_1$.

Proof: $(i), (iii)$ and (iv) follow from the Proposition 5.4 of Turull [25]. The rest follows from Lemma 3.6.4. ■

Let B be the quaternion algebra generated by ζ_1 and ζ_2 such that

$$\zeta_1^2 = -2\varepsilon_1(-1)^{\binom{n+1}{2}}, \; \zeta_2^2 = 1, \; \zeta_1\zeta_2 = -\zeta_2\zeta_1.$$

Then B represents the element 1 in $\mathrm{Br}(\mathbf{Q})$.

Let D be the quaternion algebra generated by a and b such that

$$a^2 = 2\varepsilon_1\varepsilon_2, \; b^2 = 1, \; ab = -ba.$$

Then D also represents the element 1 in $\mathrm{Br}(\mathbf{Q})$.

Let A be the quaternion algebra $(-1,-1)^{\binom{n+1}{4}}$. Then we have the following fundamental theorem:

Theorem 3.6.8 $A \otimes B \otimes C \otimes D = 1$ *in* $Br(\mathbf{Q})$. *Furthermore, the multiplicative subgroup generated by*

$$1 \otimes \frac{1}{2}\zeta_1 \otimes \gamma_i \otimes 1, \quad 1 \otimes \frac{1}{2}\zeta_1 \otimes \gamma_0 \otimes a \; (i = 1, 2, \ldots, n-1)$$

is isomorphic to $W_n[\varepsilon_1, \varepsilon_2, -1, -1, -1]$ *under the isomorphism which sends* -1 *to* -1, $1 \otimes \frac{1}{2}\zeta_1 \otimes \gamma_i \otimes 1$ *to* σ_i *and* $1 \otimes \frac{1}{2}\zeta_1 \otimes \gamma_0 \otimes a$ *to* τ *for* $i = 1, 2, \ldots, n-1$.

Proof: Since $A = (-1,-1)^{\binom{n+1}{4}} = C$ in $Br(\mathbf{Q})$, and $B = D = 1$ in $Br(\mathbf{Q})$, it follows that $A \otimes B \otimes C \otimes D = 1$ in $Br(\mathbf{Q})$. Next

$$\begin{aligned}
\sigma_i^2 &= 1 \otimes \frac{1}{4}\zeta_1^2 \otimes \gamma_i^2 \otimes 1 \\
&= 1 \otimes \frac{1}{4}[-2\varepsilon_1(-1)^{\binom{n+1}{2}}] \otimes [-2(-1)^{\binom{n+1}{2}}] \otimes 1 \\
&= \varepsilon_1[1 \otimes 1 \otimes 1 \otimes 1] \\
&= \varepsilon_1.
\end{aligned}$$

$$\begin{aligned}
\tau^2 &= 1 \otimes \frac{1}{4}\zeta_1^2 \otimes \gamma_0^2 \otimes a^2 \\
&= 1 \otimes \frac{1}{4}[-2\varepsilon_1(-1)^{\binom{n+1}{2}}][-(-1)^{\binom{n+1}{2}}] \otimes 2\varepsilon_1\varepsilon_2 \\
&= \varepsilon_2[1 \otimes 1 \otimes 1 \otimes 1] \\
&= \varepsilon_2.
\end{aligned}$$

$$\begin{aligned}
(\sigma_i\sigma_{i+1})^3 &= [1 \otimes \frac{1}{4}\zeta_1^2 \otimes \gamma_i\gamma_{i+1} \otimes 1]^3 \\
&= \varepsilon_1[1 \otimes 1 \otimes 1 \otimes 1] \\
&= \varepsilon_1,
\end{aligned}$$

And when $|i-j| \geq 2$,

$$\begin{aligned}
(\sigma_i\sigma_j)^2 &= [1 \otimes \frac{1}{4}\zeta_1^2 \otimes \gamma_i\gamma_j \otimes 1]^2 \\
&= [1 \otimes -\frac{1}{2}\varepsilon_1(-1)^{\binom{n+1}{2}} \otimes \gamma_i\gamma_j \otimes 1]^2 \\
&= 1 \otimes \frac{1}{4} \otimes (\gamma_i\gamma_j)^2 \otimes 1 \\
&= (-1)(1 \otimes 1 \otimes 1 \otimes 1) \\
&= -1.
\end{aligned}$$

When $i > 1$,

$$(\sigma_i\tau)^2 = [1 \otimes \frac{1}{4}\zeta_1^2 \otimes \gamma_i\gamma_0 \otimes a]^2$$

$$
\begin{aligned}
&= [1 \otimes -\frac{1}{2}\varepsilon_1(-1)^{\binom{n+1}{2}} \otimes \gamma_i\gamma_0 \otimes a]^2 \\
&= 1 \otimes \frac{1}{4} \otimes (\gamma_i\gamma_0)^2 \otimes a^2 \\
&= 1 \otimes \frac{1}{4} \otimes (-2) \otimes 2\varepsilon_1\varepsilon_2 \\
&= -\varepsilon_1\varepsilon_2(1 \otimes 1 \otimes 1 \otimes 1) \\
&= -\varepsilon_1\varepsilon_2.
\end{aligned}
$$

$$
\begin{aligned}
(\sigma_1\tau)^4 &= 1 \otimes \frac{1}{16} \otimes (\gamma_1\gamma_0)^4 \otimes a^4 \\
&= \frac{1}{4}(1 \otimes 1 \otimes (\gamma_1\gamma_0)^4 \otimes 1) \\
&= \frac{1}{4}(1 \otimes 1 \otimes (-4) \otimes 1) \\
&= -1.
\end{aligned}
$$

The above computations show that

$$1 \otimes \frac{1}{2}\zeta_1 \otimes \gamma_i \otimes 1, \quad 1 \otimes \frac{1}{2}\zeta_1 \otimes \gamma_0 \otimes a \ (i = 1, 2, \ldots, n-1)$$

satisfy the defining relations of $W_n[\varepsilon_1, \varepsilon_2, -1, -1, -1]$. It is not hard to show that the group generated by

$$1 \otimes \frac{1}{2}\zeta_1 \otimes \gamma_i \otimes 1, \quad 1 \otimes \frac{1}{2}\zeta_1 \otimes \gamma_0 \otimes a \ (i = 1, 2, \ldots, n-1)$$

has order at least $2|W_n|$, and hence is isomorphic to $W_n[\varepsilon_1, \varepsilon_2, -1, -1, -1]$ as desired. This completes the proof of the theorem. ∎

By above theorem, $A \otimes B \otimes C \otimes D$ is a central simple algebra representing the element $1 \in Br(\mathbf{Q})$. So we may identify $A \otimes B \otimes C \otimes D = End_{\mathbf{Q}}(P)$ for some irreducible $A \otimes B \otimes C \otimes D$-module P. Also by the above theorem, we may identify

$$\sigma_i = 1 \otimes \frac{1}{2}\zeta_1 \otimes \gamma_i \otimes 1$$

$$\tau = 1 \otimes \frac{1}{2}\zeta_1 \otimes \gamma_0 \otimes a$$

in $A \otimes B \otimes C \otimes D$ $(i = 1, 2, \ldots, n-1)$ and $W_n[\varepsilon_1, \varepsilon_2, -1, -1, -1]$ may be identified with the multiplicative subgroup of $A \otimes B \otimes C \otimes D$ generated by

$$1 \otimes \frac{1}{2}\zeta_1 \otimes \gamma_i \otimes 1, \quad 1 \otimes \frac{1}{2}\zeta_1 \otimes \gamma_0 \otimes a \ (i = 1, 2, \ldots, n-1).$$

Henceforth we use these identifications. We then view P as a $W_n[\varepsilon_1, \varepsilon_2, -1, -1, -1]$-module by restriction, and as such, let $\Im$ be the character afforded by P. The next lemma gives us the relations between $\Im$ and $\psi^{\varepsilon_2,\varepsilon_2}$.

Lemma 3.6.9 *Let G denote $W_n[\varepsilon_1, \varepsilon_2, -1, -1, -1]$ and $H = G'$ denote the distinguished normal subgroup with $G/H \simeq \mathbf{Z}_2 \times \mathbf{Z}_2$. Then we have*

$$\Im = 2(\psi^{\varepsilon_2,\varepsilon_2}|_H)^G.$$

Proof: Since the action of $A \otimes B \otimes C \otimes D$ by left multiplication on itself is isomorphic to its action on $[dim(A \otimes B \otimes C \otimes D)]^{1/2} = dim(P)$ copies of P. This enables us to calculate the trace on P of many elements of $A \otimes B \otimes C \otimes D$.

To evaluate $\Im(\omega), \omega \in W_n[\varepsilon_1, \varepsilon_2, -1, -1, -1]$, assume

$$\omega = 1 \otimes \left(\frac{1}{2}\zeta_1\right)^l \otimes \eta \otimes a^m, \eta \in C.$$

Then

$$\Im(\omega) = trace(\omega)/dim(P).$$

It is easy to see

$$\begin{aligned} trace(\omega) &= trace(1) \cdot trace\left(\frac{1}{2}\zeta_1\right)^l \cdot trace(\eta) \cdot trace(a^m) \\ &= dim(A)\xi_\emptyset(1) \cdot dim(B)\xi_\emptyset\left((\frac{1}{2}\zeta_1)^l\right) \cdot dim(C)\xi_\emptyset(\eta) \cdot dim(D)\xi_\emptyset(a^m) \\ &= 4\xi_\emptyset(1) \cdot 4\xi_\emptyset\left((\frac{1}{2}\zeta_1)^l\right) \cdot dim(C)\xi_\emptyset(\eta) \cdot 4\xi_\emptyset(a^m) \\ &= 8^2 \cdot dim(C) \cdot \xi_\emptyset\left((\frac{1}{2}\zeta_1)^l\right) \cdot \xi_\emptyset(\eta) \cdot \xi_\emptyset(a^m) \\ &= 8^2 \cdot dim(C) \cdot \xi_\emptyset(\omega), \end{aligned}$$

where $\xi_\emptyset(\omega)$ denotes the constant term of ω.

Note that

$$\xi_\emptyset\left((\frac{1}{2}\zeta_1)^l\right) \neq 0 \text{ iff } l \equiv 0\ (mod\ 2)$$

and

$$\xi_\emptyset(a^m) \neq 0 \text{ iff } m \equiv 0\ (mod\ 2).$$

In the following, we will focus on computing $\xi_\emptyset(\omega)$ with l, m even.

Let $\omega = \omega_{\alpha\beta} = \omega_1\omega_2\dots\omega_{l(\alpha)}\omega'_1\omega'_2\dots\omega'_{l(\beta)}$ be the canonical element so that $\hat{\omega}$ belongs to the class indexed by (α,β). Assume $\alpha = \left(\alpha_1 \geq \alpha_2 \geq \dots \alpha_{l(\alpha)}\right), \beta = \left(\beta_1 \geq \beta_2 \geq \dots \beta_{l(\beta)}\right)$ where

$$\omega_i = \sigma_{a_{i-1}+1}\dots\sigma_{a_i-2}\sigma_{a_i-1}\ (a_i = \alpha_1 + \dots + \alpha_i)$$

$$\omega'_i = \sigma_{b_{i-1}+1}\dots\sigma_{b_i-2}\sigma_{b_i-1}\tau_{b_i}\ (b_i = |\alpha| + \beta_1 + \dots + \beta_i)$$

and

$$\tau_j = \sigma_{j-1}^{-1}\dots\sigma_1^{-1}\tau\sigma_1\dots\sigma_{j-1}.$$

First, consider the case $\omega = \omega_i$. Then

$$\omega_i = 1 \otimes \left(\frac{1}{2}\zeta_1\right)^{\alpha_i-1} \otimes \gamma_{a_{i-1}+1}\dots\gamma_{a_i-2}\gamma_{a_i-1} \otimes 1,$$

where

$$\gamma_{a_{i-1}+1}\dots\gamma_{a_i-2}\gamma_{a_i-1} = \begin{cases} \left(\prod\xi_i\right)^{\alpha_i-1}\left[(\xi_{a_{i-1}+1}-\xi_{a_{i-1}+2})\dots(\xi_{a_i-1}-\xi_{a_i})\right] & \text{if } 2\not|n \\ (-1)^{\binom{\alpha_i}{2}}\left(\prod\xi_i\right)^{\alpha_i-1}\left[(\xi_{a_{i-1}+1}-\xi_{a_{i-1}+2})\dots(\xi_{a_i-1}-\xi_{a_i})\right] & \text{if } 2|n. \end{cases}$$

It follows $\Im(\omega_i) = 0$ unless $\alpha_i - 1 \equiv 0$ (mod 2), i.e., α_i is odd.

Since

$$\xi_\emptyset\left[(\xi_{a_{i-1}+1}-\xi_{a_{i-1}+2})\dots(\xi_{a_i-1}-\xi_{a_i})\right] = (-1)^{\frac{(\alpha_i-1)}{2}},$$

we have

$$\xi_\emptyset(\omega_i) = \begin{cases} (-1)^{\frac{(\alpha_i-1)}{2}}\left(\frac{1}{2}\zeta_1\right)^{\alpha_i-1}\left(\prod\xi_i\right)^{\alpha_i-1} & \text{if } 2\not|n \\ (-1)^{\binom{\alpha_i}{2}}(-1)^{\frac{(\alpha_i-1)}{2}}\left(\frac{1}{2}\zeta_1\right)^{\alpha_i-1}\left(\prod\xi_i\right)^{\alpha_i-1} & \text{if } 2|n. \end{cases}$$

It follows that

$$\xi_\emptyset\left(\omega_1\ldots\omega_{l(\alpha)}\right) = \prod_{i=1}^{l(\alpha)}\xi_\emptyset(\omega_i)$$

$$= \begin{cases} \prod_{i=1}^{l(\alpha)}(-1)^{\frac{(\alpha_i-1)}{2}}\left(\frac{1}{2}\zeta_1\right)^{\alpha_i-1}\left(\prod\xi_i\right)^{\alpha_i-1} & \text{if } 2\not|n \\ \prod_{i=1}^{l(\alpha)}(-1)^{\binom{\alpha_i}{2}}(-1)^{\frac{(\alpha_i-1)}{2}}\left(\frac{1}{2}\zeta_1\right)^{\alpha_i-1}\left(\prod\xi_i\right)^{\alpha_i-1} & \text{if } 2|n. \end{cases}$$

Denote $\zeta = \frac{1}{2}\zeta_1$, then $\zeta^2 = -\frac{\varepsilon_1}{2}(-1)^{\binom{(n+1)}{2}}$.

Note

$$\sum_{i=1}^{l(\alpha)}\binom{\alpha_i}{2} \equiv \frac{l(\alpha)-|\alpha|}{2}\ (mod\ 2) \text{ if } \alpha_i \equiv 1\ (mod\ 2).$$

It follows from Lemma 3.6.4 that

$$\xi_\emptyset(\omega_1\ldots\omega_{l(\alpha)}) = \begin{cases} (-1)^{\sum_{i=1}^{l(\alpha)}\frac{\alpha_i-1}{2}}\zeta^{\sum_{i=1}^{l(\alpha)}(\alpha_i-1)}\xi^{\sum_{i=1}^{l(\alpha)}(\alpha_i-1)} & \text{if } 2\not|n \\ (-1)^{\sum_{i=1}^{l(\alpha)}\binom{\alpha_i}{2}}(-1)^{\sum_{i=1}^{l(\alpha)}\frac{\alpha_i-1}{2}}\eta^{\sum_{i=1}^{l(\alpha)}(\alpha_i-1)}\xi^{\sum_{i=1}^{l(\alpha)}(\alpha_i-1)} & \text{if } 2|n. \end{cases}$$

$$= \begin{cases} (-1)^{\frac{|\alpha|-l(\alpha)}{2}}(\zeta\xi)^{|\alpha|-l(\alpha)} & \text{if } 2\not|n \\ (\zeta\xi)^{|\alpha|-l(\alpha)} & \text{if } 2|n. \end{cases}$$

For negative cycle, in general

$$\omega' = \sigma_{j+1}\ldots\sigma_{j+k-1}\tau_{j+k},$$

then

$$\sigma_{j+1}\ldots\sigma_{j+k-1} = 1\otimes\zeta^{k-1}\otimes\gamma_{j+1}\ldots\gamma_{j+k-1}\otimes 1,$$

$$\gamma_{j+1}\ldots\gamma_{j+k-1} = \begin{cases} (\xi_{j+1}-\xi_{j+2})\ldots(\xi_{j+k-1}-\xi_{j+k})\xi^{k-1} & \text{if } 2\not|n \\ (\xi_{j+1}-\xi_{j+2})\ldots(\xi_{j+k-1}-\xi_{j+k})\xi^{k-1}(-1)^{\binom{k-1}{2}} & \text{if } 2|n, \end{cases}$$

$$\begin{aligned}\tau_{j+k} &= (\varepsilon_1)^{j+k-1}\sigma_{j+k-1}\ldots\sigma_1\tau\sigma_1\ldots\sigma_{j+k-1} \\ &= \varepsilon_1^{j+k-1}(1\otimes\zeta^{2(j+k-1)}\otimes\gamma_{j+k-1}\ldots\gamma_1\gamma_0\gamma_1\ldots\gamma_{j+k-1}\otimes a),\end{aligned}$$

$$\gamma_{j+k-1}\dots\gamma_1\gamma_0\gamma_1\dots\gamma_{j+k-1} = \begin{cases} \xi^{2(j+k-1)+1}(-2)^{j+k-1}\xi_{j+k} & \text{if } 2 \not| n \\ (-1)^{j+k}\xi^{2(j+k-1)+1}(-2)^{j+k-1}\xi_{j+k} & \text{if } 2|n. \end{cases}$$

(note $\gamma_0 = \xi_1\xi$).

Therefore, if $2 \not| n$, we have

$$\begin{aligned} \xi_\emptyset(\omega') &= \xi_\emptyset(\sigma_{j+1}\dots\sigma_{j+k-1}\tau_{j+k}) \\ &= \varepsilon_1^{j+k-1}\zeta^{(k-1)+2(j+k-1)+1}(-2)^{j+k-1}\xi^{(k-1)+2(j+k-1)+1} \\ &\quad \cdot(\xi_{j+1}-\xi_{j+2})\dots(\xi_{j+k-1}-\xi_{j+k})\xi_{j+k}. \end{aligned}$$

If $2|n$, then

$$\begin{aligned} \xi_\emptyset(\omega') &= \xi_\emptyset(\sigma_{j+1}\dots\sigma_{j+k-1}\tau_{j+k}) \\ &= \varepsilon_1^{j+k-1}\zeta^{(k-1)+2(j+k-1)+1}(-2)^{j+k-1}\xi^{(k-1)+2(j+k-1)}(-1)^{\binom{k-1}{2}}(-1)^{j+k} \\ &\quad \cdot(\xi_{j+1}-\xi_{j+2})\dots(\xi_{j+k-1}-\xi_{j+k})\xi_{j+k}. \end{aligned}$$

Note that $(\xi_{j+1}-\xi_{j+2})\dots(\xi_{j+k-1}-\xi_{j+k})\xi_{j+k}$ has no constant term; in other words, it is zero unless k is even, and in that case, it is $(-1)^{\frac{k}{2}}$. Therefore, we may assume $k \equiv 0 (\text{mod } 2)$, then

$$\begin{aligned} \xi_\emptyset(\omega') &= \begin{cases} \varepsilon_1^{j+k-1}\zeta^{2j+3k-2}(-2)^{j+k-1}\xi^{2j+3k-2}(-1)^{\frac{k}{2}} & \text{if } 2 \not| n \\ (-1)^{\binom{k-1}{2}+j+k}\varepsilon_1^{j+k-1}\zeta^{2j+3k-2}(-2)^{j+k-1}\xi^{2j+3k-2}(-1)^{\frac{k}{2}} & \text{if } 2|n \end{cases} \\ &= \begin{cases} (-2\varepsilon_1)^{j+k-1}(\zeta\xi)^{2j+3k-2}(-1)^{\frac{k}{2}} & \text{if } 2 \not| n \\ (-1)^{\binom{k-1}{2}+j+k}(-2\varepsilon_1)^{j+k-1}(\zeta\xi)^{2j+3k-2}(-1)^{\frac{k}{2}} & \text{if } 2|n. \end{cases} \end{aligned}$$

It follows that if $2 \not| n$, then

$$\begin{aligned} \xi_\emptyset(\omega') &= \varepsilon_1^{j+1}(-2)^{j+k-1}(\zeta^2\xi^2)^{\frac{2j+3k-2}{2}}(-1)^{\frac{k}{2}} \\ &= \varepsilon_1^{j+1}(-2)^{j+k-1}(\varepsilon_1 2^{-1})^{\frac{2j+3k-2}{2}}(-1)^{\frac{k}{2}} \\ &= \varepsilon_1^{j+1+\frac{2j+3k-2}{2}}2^{j+k-1+\frac{2j+3k-2}{2}}(-1)^{\frac{k}{2}+j+k-1} \\ &= \varepsilon_1^{\frac{4j+3k}{2}}2^{-\frac{k}{2}}(-1)^{\frac{k}{2}+j-1} \\ &= \varepsilon_1^{\frac{k}{2}}2^{-\frac{k}{2}}(-1)^{\frac{k}{2}}. \end{aligned}$$

(since $2 \not| n, j \equiv n \ (mod\ 2)$ implies $j-1 \equiv 0 \ (mod\ 2)$).

If $2|n$, we have

$$\begin{aligned}\xi_{\emptyset}(\omega') &= (-1)^{\binom{k-1}{2}+j+k}2^{j+k-1}(-\varepsilon_1 2^{-1})^{\frac{2j+3k-2}{2}}(-1)^{\frac{k}{2}+j+k-1}\\ &= (-1)^{\binom{k-1}{2}}(\varepsilon_1)^{\frac{k}{2}}2^{-\frac{k}{2}}.\end{aligned}$$

(since $j = |\alpha| + \beta_1 + \ldots \beta_{i-1}, 2|\beta_i$, it follows $j \equiv l(\alpha) \equiv n \equiv 0$ (mod 2)).

In general, let

$$\omega' = \omega'_1 \ldots \omega'_{l(\beta)} = 1 \otimes \left(\frac{1}{2}\zeta_1\right)^l \otimes \eta \otimes a^{l(\beta)}.$$

Note $trace\left(a^{l(\beta)}\right) \neq 0$ only when $l(\beta)$ is even. In that case, $a^{l(\beta)} = (a^2)^{\frac{l(\beta)}{2}} = (2\varepsilon_1\varepsilon_2)^{\frac{l(\beta)}{2}}$. It follows that

$$\xi_{\emptyset}(\omega') = \prod_{i=1}^{l(\beta)} \xi_{\emptyset}(\omega'_i)a^{l(\beta)} \quad (j = |\alpha| + \beta_1 + \ldots \beta_{i-1}, k = \beta_i)$$

If $2 \not| n$, then

$$\begin{aligned}\xi_{\emptyset}(\omega') &= \left(\prod_{i=1}^{l(\beta)} \varepsilon_1^{\frac{\beta_i}{2}} 2^{-\frac{\beta_i}{2}}(-1)^{\frac{\beta_i}{2}}\right) a^{l(\beta)}\\ &= a^{l(\beta)}\varepsilon_1^{\sum\frac{\beta_i}{2}}2^{-\frac{1}{2}\sum\beta_i}(-1)^{\frac{1}{2}\sum\beta_i}\\ &= a^{l(\beta)}\varepsilon_1^{\frac{|\beta|}{2}}2^{-\frac{|\beta|}{2}}(-1)^{\frac{|\beta|}{2}}\\ &= (2\varepsilon_1\varepsilon_2)^{\frac{l(\beta)}{2}}\varepsilon_1^{\frac{|\beta|}{2}}2^{-\frac{|\beta|}{2}}(-1)^{\frac{|\beta|}{2}}\\ &= \varepsilon_1^{\frac{l(\beta)+|\beta|}{2}}\varepsilon_2^{\frac{l(\beta)}{2}}2^{\frac{l(\beta)-|\beta|}{2}}(-1)^{\frac{|\beta|}{2}}.\end{aligned}$$

If $2|n$, then

$$\begin{aligned}\xi_{\emptyset}(\omega') &= a^{l(\beta)}\prod_{i=1}^{l(\beta)}(-1)^{\binom{\beta_i-1}{2}}\varepsilon_1^{\frac{\beta_i}{2}}2^{-\frac{\beta_i}{2}}\\ &= a^{l(\beta)}(-1)^{\sum\binom{\beta_i-1}{2}}\varepsilon_1^{\sum\frac{\beta_i}{2}}2^{-\frac{1}{2}\sum\beta_i}\\ &= a^{l(\beta)}\varepsilon_1^{\frac{|\beta|}{2}}2^{-\frac{|\beta|}{2}}(-1)^{\frac{|\beta|}{2}}\\ &= (2\varepsilon_1\varepsilon_2)^{\frac{l(\beta)}{2}}\varepsilon_1^{\frac{|\beta|}{2}}2^{-\frac{|\beta|}{2}}(-1)^{\frac{|\beta|}{2}+l(\beta)}\\ &= \varepsilon_1^{\frac{l(\beta)+|\beta|}{2}}\varepsilon_2^{\frac{l(\beta)}{2}}2^{\frac{l(\beta)-|\beta|}{2}}(-1)^{\frac{|\beta|}{2}+l(\beta)}.\end{aligned}$$

From tha above arguments, we have $\xi_\emptyset(\omega_{\alpha\beta}) = \xi_\emptyset\left(\omega_1 \dots \omega_{l(\alpha)}\omega'_1 \dots \omega'_{l(\beta)}\right) = 0$ unless $\alpha \in OP, \beta \in EP$, where

$$\omega_{\alpha\beta} = \omega_1 \dots \omega_{l(\alpha)}\omega'_1 \dots \omega'_{l(\beta)} = 1 \otimes (\frac{1}{2}\zeta_1)^l \otimes \eta \otimes a^{l(\beta)}.$$

So for the rest of proof, we assume $\alpha \in OP, \beta \in EP$.

Recall that

$$\xi_\emptyset(\omega_1 \dots \omega_{l(\alpha)}) = \left(\frac{\varepsilon_1}{2}\right)^{\frac{|\alpha|-l(\alpha)}{2}} (-1)^{\frac{|\alpha|-l(\alpha)}{2}}.$$

If $2 \not| n$, we have

$$\begin{aligned} \xi_\emptyset\left(\omega_1 \dots \omega_{l(\alpha)}\omega'_1 \dots \omega'_{l(\beta)}\right) &= (\frac{\varepsilon_1}{2})^{\frac{|\alpha|-l(\alpha)}{2}} (-1)^{\frac{|\alpha|-l(\alpha)}{2}} \varepsilon_1^{\frac{l(\beta)+|\beta|}{2}} \varepsilon_2^{\frac{l(\beta)}{2}} 2^{\frac{l(\beta)-|\beta|}{2}} (-1)^{\frac{|\beta|}{2}} \\ &= \varepsilon_1^{\frac{n-l(\alpha)+l(\beta)}{2}} \varepsilon_2^{\frac{l(\beta)}{2}} 2^{\frac{l(\alpha)+l(\beta)-n}{2}} (-1)^{\frac{n-l(\alpha)}{2}}. \end{aligned}$$

If $2|n$, we have

$$\begin{aligned} \xi_\emptyset(\omega_1 \dots \omega_{l(\alpha)}\omega'_1 \dots \omega'_{l(\beta)}) &= (\frac{\varepsilon_1}{2})^{\frac{|\alpha|-l(\alpha)}{2}} (-1)^{\frac{|\alpha|-l(\alpha)}{2}} \varepsilon_1^{\frac{l(\beta)+|\beta|}{2}} \varepsilon_2^{\frac{l(\beta)}{2}} 2^{\frac{l(\beta)-|\beta|}{2}} (-1)^{\frac{|\beta|}{2}} \\ &= \varepsilon_1^{\frac{n-l(\alpha)+l(\beta)}{2}} \varepsilon_2^{\frac{l(\beta)}{2}} 2^{\frac{l(\alpha)+l(\beta)-n}{2}} (-1)^{\frac{n-l(\alpha)}{2}+l(\beta)}. \end{aligned}$$

It follows that

$$\begin{aligned} \Im(\alpha, \beta) &= trace(\omega_{\alpha\beta})/dim(P) \\ &= 8^2 \cdot dim(C) \cdot \xi_\emptyset(\omega_{\alpha\beta})/dim(P) \\ &= 8 \cdot dim(C)^{\frac{1}{2}} \cdot \xi_\emptyset(\omega_{\alpha\beta}) \end{aligned}$$

since $dim(P) = 8 \cdot \sqrt{dim(C)}$.

Recall that

$$dim(C) = \begin{cases} 2^n & \text{if } 2|n \\ 2^{n-1} & \text{if } 2 \not| n. \end{cases}$$

If $2 \not| n$, then

$$\begin{aligned} \Im(\alpha, \beta) &= 8 \cdot 2^{\frac{n-1}{2}} \varepsilon_1^{\frac{n-l(\alpha)+l(\beta)}{2}} \varepsilon_2^{\frac{l(\beta)}{2}} (-1)^{\frac{n-l(\alpha)}{2}} 2^{\frac{l(\alpha)+l(\beta)-n}{2}} \\ &= 8 \cdot \varepsilon_1^{\frac{n-l(\alpha)+l(\beta)}{2}} \varepsilon_2^{\frac{l(\beta)}{2}} (-1)^{\frac{n-l(\alpha)}{2}} 2^{\frac{l(\alpha)+l(\beta)-1}{2}}. \end{aligned}$$

If $2|n$, then

$$\begin{aligned}\Im(\alpha,\beta) &= 8\cdot 2^{\frac{n}{2}}\varepsilon_1^{\frac{n-l(\alpha)+l(\beta)}{2}+l(\beta)}\varepsilon_2^{\frac{l(\beta)}{2}}(-1)^{\frac{n-l(\alpha)}{2}}2^{\frac{l(\alpha)+l(\beta)-n}{2}}\\ &= 8\cdot\varepsilon_1^{\frac{n-l(\alpha)+l(\beta)}{2}}\varepsilon_2^{\frac{l(\beta)}{2}}(-1)^{\frac{n-l(\alpha)}{2}+l(\beta)}2^{\frac{l(\alpha)+l(\beta)-1}{2}}.\end{aligned}$$

We just proved the following:

If $2 \not| n$, then

$$\Im(\alpha,\beta)=\begin{cases} 8\cdot\varepsilon_1^{\frac{n-l(\alpha)+l(\beta)}{2}}\varepsilon_2^{\frac{l(\beta)}{2}}(-1)^{\frac{n-l(\alpha)}{2}}2^{\frac{l(\alpha)+l(\beta)-1}{2}} & \text{if } \alpha\in OP,\beta\in EP\\ 0 & \text{otherwise .}\end{cases}$$

If $2|n$, then

$$\Im(\alpha,\beta)=\begin{cases} 8\cdot\varepsilon_1^{\frac{n-l(\alpha)+l(\beta)}{2}}\varepsilon_2^{\frac{l(\beta)}{2}}(-1)^{\frac{n-l(\alpha)}{2}+l(\beta)}2^{\frac{l(\alpha)+l(\beta)-1}{2}} & \text{if } \alpha\in OP,\beta\in EP\\ 0 & \text{otherwise .}\end{cases}$$

This shows that the two characters $\Im$ and $2(\psi^{\varepsilon_1,\varepsilon_2}|_H)^G$ have the same values. Therefore

$$\Im = 2(\psi^{\varepsilon_1,\varepsilon_2}|_H)^G.$$

Now the proof is complete. ■

To make it simple, we will denote $\psi^{\varepsilon_1,\varepsilon_2}$ by ψ in the following study.

Lemma 3.6.10

$$(\psi|_H)^G=\begin{cases}\psi+\varepsilon\psi+\delta\psi+\varepsilon\delta\psi & \text{if } 2\not| n\\ 2(\psi+\varepsilon\psi) & \text{if } 2|n.\end{cases}$$

Proof: If $2 \not| n$ then $L(\psi)=\{1\}$. This implies $\psi|_H=\theta\in Irr(H)$. By Gallagher's Theorem of Isaacs [9], we have

$$\theta^G=\psi+\varepsilon\psi+\delta\psi+\varepsilon\delta\psi,$$

because $\{1,\varepsilon,\delta,\varepsilon\delta\}$ are all the irreducible characters of $G/H\simeq \mathbf{Z}_2\times\mathbf{Z}_2$. It follows

$$(\psi|_H)^G=\psi+\varepsilon\psi+\delta\psi+\varepsilon\delta\psi.$$

If $2|n$, then $L(\psi) = \{1, \varepsilon\delta\}$. By Theorem 3.1 of Stembridge [23], we have

$$\psi|_H = \theta_1 + \theta_2,\ \theta_1 \neq \theta_2 \in Irr(H)$$

where θ_1, θ_2 are G-conjugate. We have $\theta_1^G = \theta_2^G$. Hence $(\psi|_H)^G = 2\theta_1^G$.

Let $I = I_G(\theta_1)$ be the inertia group of θ_1, then $|G : I| = 2$ (because $\psi|_H = \theta_1 + \theta_2,\ \theta_1 \neq \theta_2 \in Irr(H)$). By Clifford's Theorem we get

$$(\theta_1^I)|_H = |I : H|\theta_1 = 2\theta_1.$$

It follows

$$[\theta_1^I, \theta_1^I]_I = [\theta_1, (\theta_1^I)|_H]_H = 2.$$

Therefore

$$\theta_1^I = \varphi_1 + \varphi_2,\ \varphi_1 \neq \varphi_2 \in Irr(I).$$

It then follows from Gallagher's Theorem 6.17 of Isaacs [9],

$$(\psi|_H)^G = 2\theta_1^G = 2(\theta_1^G) = 2(\varphi_1 + \varphi_2)^G = 2(\varphi_1^G + \varphi_2^G),$$

$\varphi_1^G, \varphi_2^G \in Irr(G)$.

Since $[\psi, (\psi|_H)^G] = [\psi|_H, \psi|_H] \neq 0$, we may assume $\varphi_1^G = \psi$. Since $\varepsilon\psi|_H = \psi|_H$, we have $[\varepsilon\psi|_H, \theta_1] \neq 0$. By 6.11 of Isaacs [9], there exists some $\chi \in Irr(I)$ such that

$$[\chi|_H, \theta_1] \neq 0, \text{and } \varepsilon\psi = \chi^G.$$

Note that

$$0 \neq [\chi_H, \theta_1] = [\chi, \theta_1^I] = [\chi, \varphi_1 + \varphi_2] = [\chi, \varphi_1] + [\chi, \varphi_2]$$

implies $\chi = \varphi_1$ or φ_2, consequently, $\chi^G = \varphi_1^G = \psi$ or $\chi^G = \varphi_2^G$. Now because $\psi \neq \varepsilon\psi, \chi^G = \varphi_2^G$, we conclude $\varphi_2^G = \varepsilon\psi$.

The proof of the lemma is now complete. ■

Corollary 3.6.11

$$\Im = \begin{cases} 2(\psi + \varepsilon\psi + \delta\psi + \varepsilon\delta\psi) & \text{if } 2 \not| n \\ 4(\psi + \varepsilon\psi) & \text{if } 2|n. \end{cases}$$

Proof: It follows from the previous two lemmas. ■

Corollary 3.6.12 *Let P be a $\mathbf{Q}G$-module affording $\Im$, $A_0(P) := End_{\mathbf{Q}G}(P)$, then*

$$dim_{\mathbf{Q}} A_0(P) = \begin{cases} 16 & \text{if } 2 \not| n \\ 32 & \text{if } 2|n. \end{cases}$$

To finish this section, we want to calculate the algebra $A_0(P)$. We have the following theorem.

Theorem 3.6.13 *Let P be a $\mathbf{Q}G$-module affording $\Im$.*

If $2 \not| n$, then

$$\begin{aligned} A_G(P) &\simeq (-1,-1)^{\binom{n+1}{4}} \otimes \langle 1\otimes <\zeta_1> \otimes 1 \otimes <a> \rangle \\ &\simeq (-1,-1)^{\binom{n+1}{4}} \otimes \mathbf{Q}\left(\sqrt{-2\varepsilon_1(-1)^{\binom{n+1}{2}}}\right) \otimes \mathbf{Q}\left(\sqrt{2\varepsilon_1\varepsilon_2}\right). \end{aligned}$$

If $2|n$, then

$$\begin{aligned} A_G(P) &\simeq (-1,-1)^{\binom{n+1}{4}} \otimes \left\langle \begin{matrix} 1\otimes <\zeta_1> \otimes 1 \otimes <a> \\ 1\otimes \zeta_2 \otimes \xi \otimes 1 \end{matrix} \right\rangle \\ &\simeq (-1,-1)^{\binom{n+1}{4}} \otimes \left(2\varepsilon_1, (-1)^{n/2}\right) \otimes \mathbf{Q}\left(\sqrt{2\varepsilon_1\varepsilon_2}\right). \end{aligned}$$

In addition, let

$$\begin{aligned} S_P^\varepsilon &= 1\otimes\zeta_2\otimes 1\otimes b \\ S_P^\delta &= 1\otimes 1\otimes 1\otimes b \end{aligned}$$

then S_P^ε is an ε-associator of P and $(S_P^\varepsilon)^2 = 1$; S_P^δ is a δ-associator of P and $(S_P^\delta)^2 = 1$. Furthermore

$$S_P^\varepsilon S_P^\delta = S_P^\delta S_P^\varepsilon.$$

Let

$$J_P = 1 \otimes 1 \otimes 1 \otimes a$$

then $J_P \in Z(A_0(P)), J_P^2 = 2\varepsilon_1\varepsilon_2 \in \mathbf{Q}^\times$, *and*

$$J_P S_P^\varepsilon = -S_P^\varepsilon J_P$$

$$J_P S_P^\delta = -S_P^\delta J_P.$$

Proof: Suppose $2 \not| n$, then $dim_{\mathbf{Q}} A_G(P) = 16$, and therefore

$$\begin{aligned} A_G(P) &\simeq A \otimes \mathbf{Q}(\zeta_1) \otimes \mathbf{Q} \otimes \mathbf{Q}(a) \\ &\simeq (-1,-1)^{\binom{n+1}{4}} \otimes \mathbf{Q}\left(\sqrt{-2\varepsilon_1(-1)^{\binom{n+1}{2}}}\right) \otimes \mathbf{Q}\left(\sqrt{2\varepsilon_1\varepsilon_2}\right), \end{aligned}$$

here

$$(-1,-1)^{\binom{n+1}{4}} = \begin{cases} (-1,-1) \ in \ Br(\mathbf{Q}) & \text{when } n \equiv 3,4,5,6 \ (mod\ 8) \\ (1,1) = 1 \in Br(\mathbf{Q}) & \text{when } n \equiv 1,2,7,8 \ (mod\ 8). \end{cases}$$

Suppose $2|n$, then $dim_{\mathbf{Q}} A_G(P) = 32$, and

$$End_{\mathbf{Q}}(P) \simeq A \otimes \langle \zeta_1, \zeta_2 \rangle \otimes C \otimes \langle a, b \rangle.$$

Let

$$\begin{aligned} \alpha &= 1 \otimes \zeta_1 \otimes 1 \otimes 1 \\ \beta &= 1 \otimes \zeta_2 \otimes \xi \otimes 1 \\ \gamma &= 1 \otimes 1 \otimes 1 \otimes a. \end{aligned}$$

Then $\alpha\beta = -\beta\alpha, \alpha, \beta, \gamma \in A_G(M), [\gamma, \langle, \alpha, \beta\rangle] = 0$. Note

$$\xi^2 = \begin{cases} (-1)^{n/2} & \text{if } 2|n \\ (-1)^{(n-1)/2} & \text{if } 2 \not| n, \end{cases}$$

it then follows from Theorem 2.8.2 that

$$\begin{aligned} A_G(P) &\simeq A \otimes \langle \alpha, \beta \rangle \otimes \langle \gamma \rangle \\ &\simeq (-1,-1)^{\binom{n+1}{4}} \otimes (\alpha^2, \beta^2) \otimes \mathbf{Q}\left(\sqrt{\gamma^2}\right) \\ &\simeq (-1,-1)^{\binom{n+1}{4}} \otimes \left(-2\varepsilon_1(-1)^{\binom{n+1}{2}}, (-1)^{n/2}\right) \otimes \mathbf{Q}\left(\sqrt{2\varepsilon_1\varepsilon_2}\right) \end{aligned}$$

Notice $(-1)^{\binom{n+1}{2}} = (-1)^{n/2}$ when $2|n$, then result follows from Proposition 2.5.1.

The remaining statements are straightforward. ■

3.7 The Algebras Associated with Double Product

In this section, we study the structures of the algebras associates with the twisted product of two modules. These results can be applied in Chapter Four to simplify the calculation of algebras associated with the spin representations of the double covers of W_n.

If M is a $\mathbf{Q}G$-module, we will use notation $A_0(M)$ to denote the algebra $A_G(M) := End_{\mathbf{Q}G}(M)$. We will also use the notation $[i,j]$ to denote the commutator of i and j, i.e.,

$$[i,j] = i^{-1}j^{-1}ij.$$

If $[i,j] = 1$, we say that i and j commute; on the other hand, if $[i,j] = -1$, we say that i and j anti-commute.

First, we like to recall the definition of associator associated with modules (refers to Section 2.4).

Let G be a group with a normal subgroup H such that $G/H \simeq \mathbf{Z}_2 \times \mathbf{Z}_2$. Recall $L = \{1, \varepsilon, \delta, \varepsilon\delta\}$ denotes the four linear representations of G with H in their kernel.

Let M be a $\mathbf{Q}G$-module affording a character χ, L can on the $\mathbf{Q}G$-modules via $M \mapsto \nu \otimes M$ $(\nu \in L)$. If $M \simeq \nu \otimes M$, we say M is self-associate (with respect to ν); otherwise, we will say that M and $\nu \otimes M$ form an associate pair (with respect to ν).

If M is self-associate, there exists an endomorphism $S \in GL_{\mathbf{Q}}(M)$ such that

$$gSv = \nu(g)Sgv$$

for all $v \in M, g \in G$. We will refer to S as an ν-associator of M. Sometimes, we use notation S_M^ν to indicate that S_M^ν is an ν-associator of M.

We use the notation $L(M) = \{\nu \in L | M \simeq \nu \otimes M\}$ to denote the stabilizer of M.

Before we calculate the algebras associated with double covers of W_n, we establish some basic results related to associators.

Lemma 3.7.1 *Let $G = W_n[\alpha]$, and H be a subgroup of G such that $\{-1, 1\} \subseteq H$ and $G/H \simeq \mathbf{Z}_2 \times \mathbf{Z}_2$. Let M be a $\mathbf{Q}W_n[\alpha]$-module. Denote $A_G(M) = End_{\mathbf{Q}G}(M), A_H(M) = End_{\mathbf{Q}H}(M)$.*

Let $A_0^b(M)$ be a basis of $A_G(M)$ over $\mathbf{Q}$. And let

$$L^a(M) = \{S_M^\nu : \nu \in L(M)\}$$

where S_M^ν is a representative of ν-associator with $S_M^\nu = 1$ if $\nu = 1$.

Then

$$L^a(M)A_0^b(M) = \{ST : S \in L^a(M), T \in A_0^b(M)\}$$

is linearly independent over $\mathbf{Q}$.

Therefore, $L^a(M)A_0^b(M)$ forms a basis of the subalgebra $< A_0(M), L^a(M) >$ of $End_{\mathbf{Q}H}(M)$. In other words, $End_{\mathbf{Q}H}(M)$ has a subalgebra generated by $A_0(M)$ and $L^a(M)$ with dimension at least

$$dim_{\mathbf{Q}} A_G(M) \times |L(M)|.$$

Proof: If $L(M) = \{1\}$, the result is obvious. For the rest of cases, since the proofs are the same, we like to take $L(M) = \{1, \varepsilon, \delta, \varepsilon\delta\}$ as an example and show that

$$L^a(M)A_0^b(M) = \{ST : S \in L^a(M), T \in A_0^b(M)\}$$

is linearly independent over $\mathbf{Q}$.

Let t_i $(i = 1, 2, 3, 4)$ be a $\mathbf{Q}$-linear combination of the elements of $A_0^b(M)$ and assume

$$S_M^1 t_1 + S_M^\varepsilon t_2 + S_M^\delta t_3 + S_M^{\varepsilon\delta} t_4 = 0. \tag{3.5}$$

To finish the proof of the lemma, we only need to show

$$t_1 = t_2 = t_3 = t_4 = 0.$$

Recall from the definition of associators, we have

$$\begin{aligned} S_M^{\varepsilon}\sigma_i &= -\sigma_i S_M^{\varepsilon}, \\ S_M^{\varepsilon}\tau_j &= \tau_j S_M^{\varepsilon}. \end{aligned}$$

$$\begin{aligned} S_M^{\delta}\sigma_i &= \sigma_i S_M^{\delta}, \\ S_M^{\delta}\tau_j &= -\tau_j S_M^{\delta}. \end{aligned}$$

$$\begin{aligned} S_M^{\varepsilon\delta}\sigma_i &= -\sigma_i S_M^{\varepsilon\delta}, \\ S_M^{\varepsilon\delta}\tau_j &= -\tau_j S_M^{\varepsilon\delta}. \end{aligned}$$

Now applying the both sides of (3.5) to σ_i, we get

$$S_M^1 t_1 - S_M^{\varepsilon} t_2 + S_M^{\delta} t_3 - S_M^{\varepsilon\delta} t_4 = 0. \tag{3.6}$$

By adding the two equations (3.5) and (3.6), we get

$$S_M^1 t_1 + S_M^{\delta} t_3 = 0. \tag{3.7}$$

Then applying the both sides of (3.7) to τ_j, we get

$$S_M^1 t_1 - S_M^{\delta} t_3 = 0. \tag{3.8}$$

By adding the two equations (3.7) and (3.8), we get

$$S_M^1 t_1 = 0. \tag{3.9}$$

It follows immediately from above equation that $t_1 = 0$.

Then (3.7) implies that $t_3 = 0$. It then follows from (3.5)

$$S_M^{\varepsilon} t_2 + S_M^{\varepsilon\delta} t_4 = 0. \tag{3.10}$$

Now applying the both sides of (3.10) to $\overset{\circ}{\tau_j}$, we get

$$S_M^{\varepsilon} t_2 - S_M^{\varepsilon\delta} t_4 = 0. \tag{3.11}$$

By adding the two equations (3.10) and (3.11), we get

$$S_M^{\varepsilon} t_2 = 0. \tag{3.12}$$

It follows immediately from above equation that $t_2 = 0$. It then follows from (3.10) that

$$S_M^{\varepsilon\delta} t_4 = 0. \tag{3.13}$$

And so finally we get $t_4 = 0$.

We just proved

$$t_1 = t_2 = t_3 = t_4 = 0$$

as we desired.

Therefore

$$L^a(M)A_0^b(M) = \{ST : S \in L^a(M), T \in A_0^b(M)\}$$

is linearly independent over $\mathbf{Q}$.

The proof is finished. ∎

Theorem 3.7.2 *Let N be a $\mathbf{Q}W_n[\alpha]$-module, M be a $\mathbf{Q}W_n[\beta]$-module, as usual, then $N \otimes M$ can be viewed as a $\mathbf{Q}W_n[\alpha\beta]$-module. If*

$$dim_{\mathbf{Q}} A_0(N \otimes M) = dim_{\mathbf{Q}} A_0(N) \times dim_{\mathbf{Q}} A_0(M) \times |L(N) \cap L(M)|$$

then

$$A_0(N \otimes M) \simeq < A_0(N) \otimes A_0(M), S_N^{\nu} \otimes S_M^{\nu} : \forall\ \nu \in L(N) \cap L(M) > .$$

In addition, $1 \otimes S_M^{\nu}$ and $S_N^{\nu} \otimes 1$ both can be chosen as ν-associators of $N \otimes M$.

Proof: Let H_α be a subgroup of $W_n[\alpha]$ such that $\{-1,1\} \subseteq H_\alpha$ and $W_n[\alpha]/H \simeq \mathbf{Z}_2 \times \mathbf{Z}_2$. The subgroup H_β of $W_n[\beta]$ and the subgroup H of $W_n[\alpha\beta]$ have similar definitions.

Let

$$T = End_{\mathbf{Q}H_\alpha}(N) \otimes End_{\mathbf{Q}H_\beta}(M) \subseteq End_{\mathbf{Q}H}(N \otimes M).$$

By the above lemma, $L^a(N)A_0^b(N)$ is linearly independent. Similarly, $L^a(M)A_0^b(M)$ is linearly independent. It then follows that

$$L^a(N)A_0^b(N) \otimes L^a(M)A_0^b(M)$$

is linearly independent over $\mathbf{Q}$.

Let

$$T_0 = \{S_N^\nu A_0^b(N) \otimes S_M^\nu A_0^b(M) : \forall \nu \in L(N) \cap L(M), \}$$

note that it is also linearly independent over $\mathbf{Q}$.

Obviously, $T_0 \subseteq A_0(N \otimes M)$, and

$$\begin{aligned} |T_0| &= dim_{\mathbf{Q}} A_0(N) \times dim_{\mathbf{Q}} A_0(M) \times |L(N) \cap L(M)| \\ &= dim_{\mathbf{Q}} A_0(N \otimes M). \end{aligned}$$

It follows that T_0 is a basis of $A_0(N \otimes M)$, and therefore

$$A_0(N \otimes M) \simeq < A_0(N) \otimes A_0(M), S_N^\nu \otimes S_M^\nu : \forall\ \nu \in L(N) \cap L(M) > .$$

The proof of the theorem is now complete. ∎

Lemma 3.7.3 *Consider a group pair $(G, H), G/H \simeq \mathbf{Z}_2$ and let $\{1, \nu\}$ be the linear representations of G/H; and they can be also regarded as linear representations of G. Assume M is a $\mathbf{Q}G$-module, and M be ν-associate, i.e., $\nu \otimes M \simeq M$ and let S_M be a ν-associator such that S_M^2 is a scalar in $\mathbf{Q}$.*

Suppose there exists an invertible $J_M \in A_0(M) := End_{\mathbf{Q}G}(M)$ such that

$$J_M S_M = -S_M J_M$$

and denote

$$C^{+}_{A_0(M)}(S_M) = \{f \in A_0(M) : fS_M = S_M f\},$$

i.e, $C^{+}_{A_0(M)}(S_M)$ *is the centralizer of* S_M *in* $A_0(M)$*, and*

$$C^{-}_{A_0(M)}(S_M) = \{f \in A_0(M) : fS_M = -S_M f\},$$

we call $C^{-}_{A_0(M)}(S_M)$ *the anti-centralizer of* S_M *in* $A_0(M)$*.*

Then

$$A_0(M) = C^{+}_{A_0(M)}(S_M) \oplus C^{-}_{A_0(M)}(S_M),$$

$$dim_{\mathbf{Q}} C^{+}_{A_0(M)}(S_M) = dim_{\mathbf{Q}} C^{-}_{A_0(M)}(S_M).$$

Furthermore we have the equations

$$\begin{aligned} J_M C^{+}_{A_0(M)}(S_M) &= C^{-}_{A_0(M)}(S_M) \\ J_M C^{-}_{A_0(M)}(S_M) &= C^{+}_{A_0(M)}(S_M). \end{aligned}$$

Proof: It follows from the definition of ν-associator that S_M can act on $A_0(M)$ by conjugation. Since S_M^2 is a scalar, S_M^2 acts on $A_0(M)$ trivially. We conclude that $A_0(M)$ is the direct sum of the two eigenspaces $C^{+}_{A_0(M)}(S_M)$ and $C^{-}_{A_0(M)}(S_M)$. Actually, $C^{+}_{A_0(M)}(S_M)$ is the eigenspace with the eigenvalue 1 and $C^{-}_{A_0(M)}(S_M)$ is the eigenspace with the eigenvalue -1. Therefore, we have

$$A_0(M) = C^{+}_{A_0(M)}(S_M) \oplus C^{-}_{A_0(M)}(S_M).$$

Next, it follows from the definitions of $J_M, C^{+}_{A_0(M)}(S_M)$, and $C^{-}_{A_0(M)}(S_M)$ that

$$\begin{aligned} J_M C^{+}_{A_0(M)}(S_M) &\subseteq C^{-}_{A_0(M)}(S_M) \\ J_M C^{-}_{A_0(M)}(S_M) &\subseteq C^{+}_{A_0(M)}(S_M). \end{aligned}$$

This forces

$$dim_{\mathbf{Q}} C^{+}_{A_0(M)}(S_M) = dim_{\mathbf{Q}} C^{-}_{A_0(M)}(S_M).$$

The lemma is now proved. ■

Theorem 3.7.4 *Let N be a $\mathbf{Q}W_n[\alpha]$-module and M be a $\mathbf{Q}W_n[\beta]$-module. Assume*

$$dim_{\mathbf{Q}} A_0(N \otimes M) = dim_{\mathbf{Q}} A_0(N) \times dim_{\mathbf{Q}} A_0(M) \times |L(N) \cap L(M)|.$$

1. Assume $L(N) \cap L(M) = \{1\}$, then

$$A_0(N \otimes M) \simeq A_0(N) \otimes A_0(M).$$

2. Assume $L(N) \cap L(M) = \{1, \nu\}$, let H_N, H_M be the kernels of ν respectively. Consider the two group pairs $(W_n[\alpha], H_N)$ and $(W_n[\beta], H_M)$.

If $A = A_{H_N}(N)$ is a **CSGA** *of even type, assume $dim_{\mathbf{Q}} A = 2 \times dim_{\mathbf{Q}} A_0$. Let*

$$\begin{aligned} A &\simeq [D, 0, d], A_0 = A_0(N) \\ Z(A_0) &\simeq \mathbf{Q} + i\mathbf{Q}, i^2 = d \in \mathbf{Q}^\times. \end{aligned}$$

where D can be chosen to be a quaternion algebra.

Let S_M be a ν-associator of M such that $S_M^2 \in \mathbf{Q}^\times$. Suppose there exists an invertible $J_M \in A_0(M)$ such that $J_M S_M = -S_M J_M$. Then

$$A_0(N \otimes M) \simeq R \otimes < 1 \otimes C_{A_0(M)}(S_M), i \otimes J_M >$$

where $R =< A_0(N) \otimes 1, S_N \otimes S_M >$ and $[R] = [A](d, S_M^2)$ in $Br(\mathbf{Q})$.

If $A = A_{H_N}(N)$ is a **CSGA** *of odd type, let*

$$\begin{aligned} A &\simeq [D, 1, d], A_0 = A_0(N) \\ Z(A) &\simeq \mathbf{Q} + S_N\mathbf{Q}, S_N^2 = d \in \mathbf{Q}^\times. \end{aligned}$$

Note that S_N is a ν-associator of N.

Let S_M be a ν-associator of M. Then

$$A_0(N \otimes M) \simeq A_0(N) \otimes < S_N \otimes S_M, 1 \otimes A_0(M) > .$$

3. Assume $L(N) \cap L(M) = \{1, \varepsilon, \delta, \varepsilon\delta\}$, then

$$A_0(N \otimes M) =< A_0(N) \otimes A_0(M), S_N^\varepsilon \otimes S_M^\varepsilon, S_N^\delta \otimes S_M^\delta > .$$

Furthermore suppose $A_0(N)$ is a central simple algebra and associators $S_N^\varepsilon, S_N^\delta$ can be chosen such that

$$[A_0(N), S_N^\varepsilon] = 1 = [A_0(N), S_N^\delta].$$

Then

$$A_0(N \otimes M) \simeq A_0(N) \otimes < 1 \otimes A_0(M), S_N^\varepsilon \otimes S_M^\varepsilon, S_N^\delta \otimes S_M^\delta > .$$

In addition, assume $A_0(M)$ is a central simple algebra and associators $S_M^\varepsilon, S_M^\delta$ can be chosen such that

$$[A_0(M), S_M^\varepsilon] = 1 = [A_0(M), S_M^\delta],$$

then

$$A_0(N \otimes M) \simeq A_0(N) \otimes A_0(M) \otimes < S_N^\varepsilon \otimes S_M^\varepsilon, S_N^\delta \otimes S_M^\delta > .$$

Here $< S_N^\varepsilon \otimes S_M^\varepsilon, S_N^\delta \otimes S_M^\delta >$ has one of the following structures:

(i) it is a quaternion algebra $\left((S_N^\varepsilon)^2(S_M^\varepsilon)^2, (S_N^\delta)^2(S_M^\delta)^2\right)$, or

(ii) it is a direct sum of fields and each summand is isomorphic to

$$\mathbf{Q}\left(\sqrt{(S_N^\varepsilon)^2(S_M^\varepsilon)^2}, \sqrt{(S_N^\delta)^2(S_M^\delta)^2}\right).$$

Proof: We divide the proof into several cases:

1. Assume $L(N) \cap L(M) = 1$, then the result is obvious.

2. Assume $L(N) \cap L(M) = \{1, \nu\}$. If first A is a **CSGA** of even type. By Lemma 3.7.3

$$dim_{\mathbf{Q}} A_0(M) = 2 \times dim_{\mathbf{Q}} C_{A_0(M)}(S_M).$$

By Lemma 2.5.5 there exists some ν-associator $S_N \in A_1$ such that $iS_N = -S_N i$ and $< i, S_N >$ is a quaternion algebra.

Let $R =< A_0(N) \otimes 1, S_N \otimes S_M >$. By Theorem 2.5.6 R is a central simple algebra. Notice $< i \otimes J_M, 1 \otimes C_{A_0(M)}(S_M) \subseteq C_{A_0(N \otimes M)}(R)$. By a dimension count

and Theorem 2.8.2, we get

$$A_0(N \otimes M) \simeq R \otimes < i \otimes J_M, 1 \otimes C_{A_0(M)}(S_M) > .$$

By Theorem 2.5.6, we have

$$[R] = [A](i^2, S_M^2) = [A](d, S_M^2)$$

in $Br(\mathbf{Q})$.

If next A is a **CSGA** of odd type, since $A_0(N)$ is a central simple algebra and $< 1 \otimes A_0(M), S_N \otimes S_M > \subseteq C_{A_0(N\otimes M)}(A_0(N))$, by a dimension count and Theorem 2.8.2, we get

$$A_0(N \otimes M) \simeq A_0(N) \otimes < 1 \otimes A_0(M), S_N \otimes S_M > .$$

3. Since $A_0(N)$ is a central simple algebra and $< 1 \otimes A_0(M), S_N^\varepsilon \otimes S_M^\varepsilon, S_N^\delta \otimes S_M^\delta > \subseteq C_{A_0(N\otimes M)}(A_0(N))$, by a dimension count and Theorem 2.8.2, we get

$$A_0(N \otimes M) \simeq A_0(N) \otimes < 1 \otimes A_0(M), S_N^\varepsilon \otimes S_M^\varepsilon, S_N^\delta \otimes S_M^\delta >$$

Additionally, under the condition

$$[A_0(M), S_M^\varepsilon] = 1 = [A_0(M), S_M^\delta]$$

by the same argument, we have

$$< 1 \otimes A_0(M), S_N^\varepsilon \otimes S_M^\varepsilon, S_N^\delta \otimes S_M^\delta > \simeq A_0(M) \otimes < S_N^\varepsilon \otimes S_M^\varepsilon, S_N^\delta \otimes S_M^\delta >,$$

therefore, we get

$$A_0(N \otimes M) \simeq A_0(N) \otimes A_0(M) \otimes < S_N^\varepsilon \otimes S_M^\varepsilon, S_N^\delta \otimes S_M^\delta > .$$

Finally, if $S_N^\varepsilon \otimes S_M^\varepsilon$ and $S_N^\delta \otimes S_M^\delta$ anti-commute, they generate a quaternion algebra as we claimed. If $S_N^\varepsilon \otimes S_M^\varepsilon$ and $S_N^\delta \otimes S_M^\delta$ commute, they generate a 4-dimensional abelian $\mathbf{Q}$-algebra. Then our desired result follows from the Corollary 2.8.5.

Therefore the theorem is now completely proved. ∎

Corollary 3.7.5 *Assume that*

$$dim_{\mathbf{Q}}A_0(N\otimes M) = dim_{\mathbf{Q}}A_0(N) \times dim_{\mathbf{Q}}A_0(M) \times |L(N)\cap L(M)|$$

and $L(N)\cap L(M) = \{1,\nu\}$, *and also that* A *is a* **CSGA** *of odd type, and a* ν-*associator* S_M *can be chosen such that* $S_M^2 \in \mathbf{Q}^\times$.

If there exits $J_M \in Z(A_0(M))$ *such that* $J_M S_M = -S_M J_M$ *and* $J_M^2 \in \mathbf{Q}^\times$, *then*

$$A_0(N\otimes M) \simeq A_0(N)\otimes (J_M^2, S_N^2 S_M^2)\otimes C_{A_0(M)}(S_M).$$

Proof: By assumption $< J_M, S_M >$ generates a quaternion algebra (J_M^2, S_M^2). By Lemma 3.7.3, we get

$$dim_{\mathbf{Q}}A_0(M) = 2\times dim_Q C_{A_0(M)}(S_M).$$

Then by a dimension count and Theorem 2.8.2 we get

$$< 1\otimes A_0(M), S_N\otimes S_M > \simeq < 1\otimes J_M, S_N\otimes S_M > \otimes \left(1\otimes C_{A_0(M)}(S_M)\right).$$

As the result, we have

$$< 1\otimes A_0(M), S_N\otimes S_M > \simeq (J_M^2, S_N^2 S_M^2)\otimes C_{A_0(M)}(S_M).$$

Consequently

$$A_0(N\otimes M) \simeq A_0(N)\otimes (J_M^2, S_N^2 S_M^2)\otimes C_{A_0(M)}(S_M).$$

■

Corollary 3.7.6 *Assume that*

$$dim_{\mathbf{Q}}A_0(N\otimes M) = dim_{\mathbf{Q}}A_0(N) \times dim_{\mathbf{Q}}A_0(M) \times |L(N)\cap L(M)|$$

and $L(N)\cap L(M) = \{1,\nu\}$, *and also* A *is a* **CSGA** *of odd type and a* ν-*associator* S_M *can be chosen such that* $S_M^2 \in \mathbf{Q}^\times$.

Consider the group pair $(W_n[\beta], H_\beta)$ and assume $B = End_{\mathbf{Q}H_\beta}(M)$ is a **CSGA**.

If B is of even type, assume $dim_{\mathbf{Q}}B = 2 \times dim_{\mathbf{Q}}B_0$, and let

$$\begin{aligned} B &\simeq [E, 0, b], B_0 = B_0(M) \\ Z(B_0) &\simeq \mathbf{Q} + j\mathbf{Q}, j^2 = b \in \mathbf{Q}^\times. \end{aligned}$$

where E can be chosen to be a quaternion algebra.

Then we have

$$A_0(N \otimes M) \simeq A_0(N)[B](b, S_N^2)$$

in $Br(\mathbf{Q})$.

If B is of odd type, let

$$\begin{aligned} B &\simeq [E, 1, b], B_0 = B_0(M), \\ Z(B) &\simeq \mathbf{Q} + S_M\mathbf{Q}, S_M^2 = b \in \mathbf{Q}^\times, \end{aligned}$$

where S_M can be chosen to be a ν-associator.

Then

$$A_0(N \otimes M) \simeq A_0(N) \otimes A_0(M) \otimes \mathbf{Q}\left(\sqrt{S_N^2 S_M^2}\right).$$

Proof: If B is of even type, by Theorem 2.5.6

$$< 1 \otimes A_0(M), S_N \otimes S_M >= [B](b, S_N^2)$$

therefore

$$A_0(N \otimes M) = A_0(N)[B](b, S_N^2)$$

in $Br(\mathbf{Q})$.

If B is of odd type, since $A_0(M)$ is a central simple algebra and $[A_0(M), S_M] = 1$, by a dimension count and Theorem 2.8.2, we get

$$\langle 1 \otimes A_0(M), S_N \otimes S_M \rangle \simeq A_0(M) \otimes \mathbf{Q}\left(\sqrt{S_N^2 S_M^2}\right),$$

therefore

$$A_0(N \otimes M) = A_0(N) \otimes A_0(M) \otimes \mathbf{Q}\left(\sqrt{S_N^2 S_M^2}\right).$$

The proof is finished. ■

The following theorem will deal with the case when one of the modules N and M affords $\Theta^{\varepsilon_1,\varepsilon_2}$.

Theorem 3.7.7 *Let N be a $\mathbf{Q}W_n[\alpha]$-module affording $\theta^{\varepsilon_1,\varepsilon_2}$, and let M be a $\mathbf{Q}W_n[\beta]$-module. Consider the twisted tensor product $N \otimes M$. Recall the definition of $\Theta^{\varepsilon_1,\varepsilon_2}$ in Section 3.3 that N has y, x, and xy as its $\varepsilon, \delta, \varepsilon\delta$ associators respectively. Assume*

$$dim_{\mathbf{Q}} A_0(N \otimes M) = dim_{\mathbf{Q}} A_0(N) \times dim_{\mathbf{Q}} A_0(M) \times |L(N) \cap L(M)|.$$

1. Assume $L(M) = \{1\}$, then

$$A_0(N \otimes M) \simeq (\varepsilon_1, \varepsilon_2) \otimes A_0(M).$$

2. Assume $L(M) = \{1, \nu\}$, let H be the kernel of ν. Consider the group pair $(W_n[\beta], H)$ and assume the graded algebra $A = End_{\mathbf{Q}H}(M)$ is a **CSGA** *with $A_0 = End_{\mathbf{Q}G}(M)$ where $G = W_n[\beta]$.*

If A is of even type, assume $dim_{\mathbf{Q}} A = 2 \times dim_{\mathbf{Q}} A_0$, and let

$$\begin{aligned} A &\simeq [D, 0, d], D = [A] \\ Z(A_0) &\simeq \mathbf{Q} + i\mathbf{Q}, i^2 = d \in \mathbf{Q}^{\times}, \end{aligned}$$

where D can be chosen to be a quaternion algebra.

Then

$$A_0(N \otimes M) \simeq (\varepsilon_1, \varepsilon_2) A_H(M) \left(d, (S_N^{\nu})^2\right)$$

in $Br(\mathbf{Q})$.

If A is of odd type,

$$\begin{aligned} A &\simeq [D, 1, d], D = [A] \\ Z(A) &\simeq \mathbf{Q} + i\mathbf{Q}, i \in A_1, i^2 = d \in \mathbf{Q}^{\times}, \end{aligned}$$

where i is a ν-associator.

Then

$$A_0(N \otimes M) \simeq (\varepsilon_1, \varepsilon_2) \otimes A_0(M) \otimes \mathbf{Q}\left(\sqrt{d(S_N^\nu)^2}\right).$$

3. Assume $L(M) = \{1, \varepsilon, \delta, \varepsilon\delta\}$, then

$$A_0(N \otimes M) \simeq (\varepsilon_1, \varepsilon_2) \otimes < 1 \otimes A_0(M), y \otimes S_M^\varepsilon, x \otimes S_M^\delta > .$$

Additionally, if $S_M^\varepsilon S_M^\delta = S_M^\delta S_M^\varepsilon$, $A_0(M)$ is a central simple algebra, and

$$[A_0(M), S_M^\varepsilon] = 1 = [A_0(M), S_M^\delta],$$

then

$$A_0(N \otimes M) \simeq (\varepsilon_1, \varepsilon_2) \otimes A_0(M) \otimes \left(\varepsilon_2 (S_M^\varepsilon)^2, \varepsilon_1 (S_M^\delta)^2\right).$$

On the other hand, if $S_M^\varepsilon S_M^\delta = -S_M^\delta S_M^\varepsilon$, $A_0(M)$ is a central simple algebra and

$$[A_0(M), S_M^\varepsilon] = 1 = [A_0(M), S_M^\delta]$$

then

$$A_0(N \otimes M) \simeq (\varepsilon_1, \varepsilon_2) \otimes A_0(M) \otimes R$$

where R is a 4-dimensional abelian $\mathbf{Q}$-algebra which is a direct sum of fields and each of the summands is isomorphic to

$$\mathbf{Q}\left(\sqrt{\varepsilon_2 (S_M^\varepsilon)^2}, \sqrt{\varepsilon_1 (S_M^\delta)^2}\right).$$

Proof: Notice by Proposition 3.3.3, $A_0(N) \simeq (\varepsilon_1, \varepsilon_2)$ and $[L(N), y] = 1 = [L(N), x]$.

If $|L(M)| = 1$, then the result follows from Theorem 3.7.4.

If $|L(M)| = 2$, then the result follows from Corollary 3.7.6.

If $|L(M)| = 4$, since $xy = -yx$ then the result follows from Theorem 3.7.4.

■

Next we consider the case when one of the two modules N and M affords $\rho^{\varepsilon_1,\varepsilon_2}$.

Theorem 3.7.8 *Let N be a $\mathbf{Q}W_n[\alpha]$-module affording $\rho^{\varepsilon_1,\varepsilon_2}$ with $\alpha = [\varepsilon_1, \varepsilon_2, 1, 1, 1]$. Let M be a $\mathbf{Q}W_n[\beta]$-module. Assume*

$$dim_{\mathbf{Q}} A_0(N \otimes M) = dim_{\mathbf{Q}} A_0(N) \times dim_{\mathbf{Q}} A_0(M) \times |L(N) \cap L(M)|.$$

1. If $L(M) = \{1\}$, then

$$A_0(N \otimes M) \simeq A_0(N) \otimes A_0(M),$$

where

$$A_0(N) \simeq \begin{cases} \mathbf{Q} \oplus \mathbf{Q} \oplus \mathbf{Q} \oplus \mathbf{Q} & \text{if } \varepsilon_1 = 1 = \varepsilon_2 \\ \mathbf{Q}(\sqrt{\varepsilon_1}, \sqrt{\varepsilon_2}) \oplus \mathbf{Q}(\sqrt{\varepsilon_1}, \sqrt{\varepsilon_2}) & \text{otherwise .} \end{cases}$$

2. If $L(M) = \{1, \nu\}$, let H be the kernel of ν. Consider the group pair (G, H) with $G = W_n[\beta]$ and the graded algebra $A = A_H(M) = End_{\mathbf{Q}H}(M)$ with $A_0 = End_{\mathbf{Q}G}(M)$.

*Suppose A is a **CSGA** of even type, assume $dim_{\mathbf{Q}} A = 2 \times dim_{\mathbf{Q}} A_0$, and let*

$$\begin{aligned} A &\simeq [D, 0, d], A_0 = End_{\mathbf{Q}G}(M) \\ Z(A_0) &\simeq \mathbf{Q} + i\mathbf{Q}, i^2 = d \in \mathbf{Q}^{\times}, \end{aligned}$$

where D can be chosen to be a quaternion algebra.

Then

$$A_0(N \otimes M) \simeq A_H \otimes R$$

where R is a direct sum of fields; let R_0 be any of its summands.

If $\nu = \varepsilon$, then $R_0 \simeq \mathbf{Q}\left(\sqrt{d\varepsilon_1}, \sqrt{\varepsilon_2}\right)$.

If $\nu = \delta$, then $R_0 \simeq \mathbf{Q}\left(\sqrt{d\varepsilon_2}, \sqrt{\varepsilon_1}\right)$.

If $\nu = \varepsilon\delta$, then $R_0 \simeq \mathbf{Q}\left(\sqrt{d\varepsilon_1}, \sqrt{\varepsilon_1\varepsilon_2}\right)$.

Suppose A is a **CSGA** *of odd type, let*

$$
\begin{aligned}
A &\simeq [D,1,d], A_0 = End_{\mathbf{Q}G}(M) \\
Z(A) &\simeq \mathbf{Q} + i\mathbf{Q}, i^2 = d \in \mathbf{Q}^{\times}.
\end{aligned}
$$

Note that i is a ν-associator of M.

If $\nu = \varepsilon$, then

$$A_0(N \otimes M) \simeq A_0(M) \otimes (d, \varepsilon_1) \otimes \mathbf{Q}(\sqrt{\varepsilon_2}).$$

If $\nu = \delta$, then

$$A_0(N \otimes M) \simeq A_0(M) \otimes (d, \varepsilon_2) \otimes \mathbf{Q}(\sqrt{\varepsilon_1}).$$

If $\nu = \varepsilon\delta$, then

$$A_0(N \otimes M) \simeq A_0(M) \otimes (d, \varepsilon_2) \otimes \mathbf{Q}(\sqrt{\varepsilon_1 \varepsilon_2}).$$

3. Assume $L(M) = \{1, \varepsilon, \delta, \varepsilon\delta\}$.

Let H_1 be the kernel of ε and H_2 be the kernel of δ. Consider the two group pairs $(G, H_1), (G, H_2)$.

Consider the two graded algebras $A = A_{H_1}(M)$ and $B = A_{H_2}(M)$. Assume both A and B are **CSGA** *of odd type. Let*

$$
\begin{aligned}
A &\simeq [D,1,d], A_0 = End_{\mathbf{Q}G}(M) \\
Z(A) &\simeq \mathbf{Q} + i\mathbf{Q}, i^2 = d_1 \in \mathbf{Q}^{\times}
\end{aligned}
$$

where i can be chosen to be an ε-associator.

Similarly

$$
\begin{aligned}
B &\simeq [E,1,d_2], B_0 = End_{\mathbf{Q}G}(M) \\
Z(B) &\simeq \mathbf{Q} + j\mathbf{Q}, j^2 = d_2 \in \mathbf{Q}^{\times}
\end{aligned}
$$

where j can be chosen to be a δ-associator.

If $ij = -ji$, *then*

$$A_0(N \otimes M) \simeq A_0(M) \otimes (d_1\varepsilon_2, \varepsilon_1) \otimes (\varepsilon_2, d_2).$$

If $ij = ji$, *then*

$$A_0(N \otimes M) \simeq A_0(M) \otimes (\varepsilon_1, d_1) \otimes (\varepsilon_2, d_2).$$

Proof: Recall from Theorem 3.4.2

$$A_0(N) =< \mathcal{X}, \mathcal{Y} >, \mathcal{X}^2 = \varepsilon_1, \mathcal{Y}^2 = \varepsilon_2, \mathcal{X}\mathcal{Y} = \mathcal{Y}\mathcal{X},$$

with $\mathcal{A}$ an ε-associator and $\mathcal{B}$ a δ-associator such that $\mathcal{A}\mathcal{B} = \mathcal{B}\mathcal{A}$ and

$$\begin{aligned} \mathcal{A}\mathcal{X} &= -\mathcal{X}\mathcal{A}, \mathcal{A}\mathcal{Y} = \mathcal{Y}\mathcal{A}; \\ \mathcal{B}\mathcal{X} &= \mathcal{X}\mathcal{B}, \mathcal{B}\mathcal{Y} = -\mathcal{Y}\mathcal{B}. \end{aligned}$$

1. Assume $L(M) = 1$.

It then follows from Theorem 3.7.4 that

$$A_0(N \otimes M) \simeq A_0(N) \otimes A_0(M).$$

The structure of $A_0(N)$ comes from Theorem 3.4.2.

2. Assume next $|L(M)| = 2$.

First consider the case $L(M) = \{1, \varepsilon\}$.

Let H be the kernel of ε and consider the group pair $(W_n[\beta], H)$ and the graded algebra $A = End_{\mathbf{Q}H}(M)$ with $A_0 = End_{\mathbf{Q}W_n[\beta]}(M)$.

If first A is a **CSGA** of even type. Let

$$\begin{aligned} A &\simeq [D, 0, d], A_0 = A_G(M) \\ Z(A_0) &\simeq \mathbf{Q} + i\mathbf{Q}, i^2 = d \in \mathbf{Q}^\times. \end{aligned}$$

By Theorem 3.7.4, we get

$$A_0(N \otimes M) \simeq< 1 \otimes A_0(M), S_N \otimes S_M > \otimes < J_N \otimes i, C_{A_0(N)}(S_N) \otimes 1 >$$

with $S_N = \mathcal{A}, J_N = \mathcal{X}, C_{A_0(N)}(S_N) =< \mathcal{Y} >$.

Since $S_N^2 = \mathcal{A}^2 = 1$, it then follows from Corollary 2.5.7 that

$$< 1 \otimes A_0(M), S_N \otimes S_M > \simeq A.$$

Now $< J_N \otimes i, C_{A_0(N)}(S_N) \otimes 1 >=< \mathcal{X} \otimes i, \mathcal{Y} \otimes 1 >$.

We conclude from above facts that

$$A_0(N \otimes M) \simeq A \otimes < \mathcal{X} \otimes i, \mathcal{Y} \otimes 1 >$$

where $< \mathcal{X} \otimes i, \mathcal{Y} \otimes 1 >$ is a 4-dimensional abelian algebra which is a direct sum of fields and each of its summands is isomorphic to

$$\mathbf{Q}\left(\sqrt{\mathcal{X}^2 i^2}, \sqrt{\mathcal{Y}^2}\right) \simeq \mathbf{Q}\left(\sqrt{d\varepsilon_1}, \sqrt{\varepsilon_2}\right).$$

The proofs are similar in the cases when $L(M) = \{1, \delta\}$ or $L(M) = \{1, \varepsilon\delta\}$.

If next A is a **CSGA** of odd type, let

$$\begin{aligned} A &\simeq [D, 1, d], A_0 = A_0(M) \\ Z(A) &\simeq \mathbf{Q} + S_M \mathbf{Q}, S_M^2 = d \in \mathbf{Q}^{\times}. \end{aligned}$$

where S_M can be chosen to be an ε-associator.

Let $J_N = \mathcal{X} \in A_0(N)$, then $\mathcal{X}\mathcal{A} = -\mathcal{A}\mathcal{X}$, by Corollary 3.7.5, we get

$$A_0(N \otimes M) \simeq A_0(M) \otimes (\mathcal{X}^2, S_N^2 S_M^2) \otimes C_{A_0(N)}(S_N)$$

with $S_N = \mathcal{A}$.

It is easy to get $C_{A_0(N)}(S_N) =< \mathcal{Y} >$. It follows then

$$A_0(N \otimes M) \simeq A_0(M) \otimes (d, \varepsilon_1) \otimes \mathbf{Q}(\sqrt{\varepsilon_2}).$$

The arguments are similar in the cases when $L(M) = \{1, \delta\}$ or $L(M) = \{1, \varepsilon\delta\}$.

This then finishes the proof of the case $|L(M)| = 2$.

3. $|L(M)| = 4$, i.e, $L(M) = \{1, \varepsilon, \delta, \varepsilon\delta\}$.

If $ij = -ji$, let

$$\begin{aligned}
\alpha &= 1 \otimes A_0(M) \\
\beta &= \mathcal{A}\mathcal{Y} \otimes i \\
\gamma &= \mathcal{X} \otimes 1 \\
\omega &= \mathcal{Y} \otimes 1 \\
\pi &= \mathcal{B} \otimes j.
\end{aligned}$$

Then we have

$$\begin{aligned}
A_0(N \otimes M) &\simeq <\alpha> \otimes <\beta, \gamma> \otimes <\omega, \pi> \\
&\simeq A_0(M) \otimes (\beta^2, \gamma^2) \otimes (\omega^2, \pi^2) \\
&\simeq A_0(M) \otimes (\mathcal{A}^2\mathcal{Y}^2 i^2, \mathcal{X}^2) \otimes (\mathcal{Y}^2, \mathcal{B}^2 j^2) \\
&\simeq A_0(M) \otimes (d_1\varepsilon_2, \varepsilon_1) \otimes (\varepsilon_2, d_2).
\end{aligned}$$

If $ij = ji$, let

$$\begin{aligned}
\alpha &= 1 \otimes A_0(M) \\
\beta &= \mathcal{A} \otimes i \\
\gamma &= \mathcal{X} \otimes 1 \\
\omega &= \mathcal{Y} \otimes 1 \\
\pi &= \mathcal{B} \otimes j.
\end{aligned}$$

Then we have

$$\begin{aligned}
A_0(N \otimes M) &\simeq <\alpha> \otimes <\beta, \gamma> \otimes <\omega, \pi> \\
&\simeq A_0(M) \otimes (\beta^2, i^2) \otimes (\omega^2, \pi^2) \\
&\simeq A_0(M) \otimes (\mathcal{X}^2, i^2\mathcal{A}^2) \otimes (\mathcal{Y}^2, j^2\mathcal{B}^2) \\
&\simeq A_0(M) \otimes (d_1, \varepsilon_1) \otimes (\varepsilon_2, d_2).
\end{aligned}$$

The proof of the theorem is now complete. ∎

3.8 The Algebras Associated with Triple Product

In this section, we will study the structures of algebras associated with the twisted product of three modules. We are not actually interested in the general modules, we only focus on the specific twisted product $\chi\theta_0\psi^{\varepsilon_1,\varepsilon_2}$, where χ is either $X^{\lambda,\mu}$ or $\varphi^{\lambda,\mu}$.

Let O be a $\mathbf{Q}W_n[\alpha]$-module affording θ_0, P a $\mathbf{Q}W_n[\beta]$-module affording $\Im$ and N a $\mathbf{Q}W_n[\gamma]$-module affording $X^{\lambda,\mu}$ (if $\chi = X^{\lambda,\mu}$); $r\varphi^{\lambda,\mu}$ if $\lambda, \mu \in DP^+$, or DP^-; $r(\varphi^{\lambda,\mu} + \varepsilon\varphi^{\lambda,\mu})$ otherwise (if $\chi = \varphi^{\lambda,\mu}$), where r is some positive integer. Define $M = O \otimes P$ to be the twisted product of O and P. Then we consider the twisted product $W := N \otimes O \otimes P$. We want to study the structure of $A_0(W) := End_{\mathbf{Q}G}(W)$ where G is the twisted product of the three groups $W_n[\alpha], W_n[\beta]$ and $W_n[\gamma]$.

In this section, we will frequently use all these notations N, O, P and M without further mention unless otherwise stated.

Recall that

$$\Im = \begin{cases} 2(\psi + \varepsilon\psi + \delta\psi + \varepsilon\delta\psi) & \text{if } 2 \not| n \\ 4(\psi + \varepsilon\psi) & \text{if } 2|n. \end{cases}$$

where ψ is an abbreviation of $\psi^{\varepsilon_1,\varepsilon_2}$.

We need the following lemmas for our purpose.

Lemma 3.8.1

$$\begin{array}{rcl} \varepsilon(\psi X^{\lambda,\emptyset}) & = & \psi X^{\lambda,\emptyset} \text{ iff } \lambda \in SC, \\ \varepsilon\delta(\psi X^{\lambda,\emptyset}) & = & \psi X^{\lambda,\emptyset} \text{ iff } n \text{ is even }, \\ \delta(\psi X^{\lambda,\emptyset}) & = & \psi X^{\lambda,\emptyset} \text{ iff } \lambda \in SC \text{ and } n \text{ is even iff } L(\psi X^{\lambda,\emptyset}) = \{1, \varepsilon, \delta, \varepsilon\delta\} \end{array}$$

Proof: Refers to page 449-450 of Stembridge [23]. ■

Lemma 3.8.2 *Denote $\psi^{\varepsilon_1,\varepsilon_2}$ by ψ.*

$$\begin{array}{rcl} \varepsilon(\psi\varphi^{\lambda,\emptyset}) & = & \psi\varphi^{\lambda,\emptyset} \text{ iff } \lambda \in DP^+, \\ \varepsilon\delta(\psi\varphi^{\lambda,\emptyset}) & = & \psi\varphi^{\lambda,\emptyset} \text{ iff } n \text{ is even }, \\ \delta(\psi\varphi^{\lambda,\emptyset}) & = & \psi\varphi^{\lambda,\emptyset} \text{ iff } \lambda \in DP^+ \text{ and } n \text{ is even iff } L(\psi\varphi^{\lambda,\emptyset}) = \{1, \varepsilon, \delta, \varepsilon\delta\}. \end{array}$$

Proof: Refers to page 449-450 of Stembridge [23]. ■

Lemma 3.8.3 *With the above notations, let N be a $\mathbf{Q}W_n[\gamma]$-module affording $X^{\lambda,\emptyset}$ and $W = N \otimes M = N \otimes O \otimes P$, then*

$$dim_{\mathbf{Q}} A_0(W) = dim_{\mathbf{Q}} A_0(N) \times dim_{\mathbf{Q}} A_0(M) \times |L(N) \cap L(M)|,$$

and

$$dim_{\mathbf{Q}} A_0(M) = dim_{\mathbf{Q}} A_0(O) \times dim_{\mathbf{Q}} A_0(P) \times |L(O) \cap L(P)|,$$

Proof: By assumption W affords $8\theta_0\psi X^{\lambda,\emptyset}$. Notice that $\psi X^{\lambda,\emptyset}$ is irreducible (see 9.2 of Stembridge [23]), so by Proposition 3.3.4 we have

$$\begin{aligned} dim_{\mathbf{Q}} A_G(W) &= \|8\theta_0\psi X^{\lambda,\emptyset}\|^2 \\ &= 8^2\|\theta_0\psi X^{\lambda,\emptyset}\|^2 \\ &= 8^2|L(\psi X^{\lambda,\emptyset})| \\ &= \begin{cases} 64 & \text{if } \lambda \notin SC \text{ and } 2 \not| n \\ 64 \times 2 & \text{if } \lambda \notin SC \text{ and } 2|n \\ 64 \times 4 & \text{if } \lambda \in SC \text{ and } 2|n. \end{cases} \end{aligned}$$

The last equality follows from Lemma 3.8.1.

Similarly, M affords $\theta_0\Im = 8\theta_0\psi$, by Proposition 3.3.4 we have

$$\begin{aligned} dim_{\mathbf{Q}} A_G(M) &= \|8\theta_0\psi\|^2 \\ &= 8^2\|\theta_0\psi\|^2 \\ &= 8^2|L(\psi)| \\ &= \begin{cases} 8^2 & \text{if } 2 \not| n \\ 8^2 \times 2 & \text{if } 2|n. \end{cases} \end{aligned}$$

The last equality follows from Corollary 3.6.6.

Notice $dim_{\mathbf{Q}} A_0(O) = 4$ since $A_0(O) \simeq (1,1)$, $dim_{\mathbf{Q}} A_0(N) = 1$ since $A_0(O) \simeq \mathbf{Q}$, and

$$dim_{\mathbf{Q}} A_0(P) = \begin{cases} 4^2 & \text{if } 2 \not| n \\ 4^2 \times 2 & \text{if } 2|n. \end{cases}$$

Also notice

$$L(N) = \begin{cases} \{1\} & \text{if } \lambda \notin SC \\ \{1, \varepsilon\} & \text{if } \lambda \in SC \end{cases}$$

as well as $L(M) = L(O) \cap L(P) = \{1, \varepsilon, \delta, \varepsilon\delta\}$.

It then is straightforward to check that

$$dim_{\mathbf{Q}} A_0(W) = dim_{\mathbf{Q}} A_0(N) \times dim_{\mathbf{Q}} A_0(M) \times |L(N) \cap L(M)|,$$

and

$$dim_{\mathbf{Q}} A_0(M) = dim_{\mathbf{Q}} A_0(O) \times dim_{\mathbf{Q}} A_0(P) \times |L(O) \cap L(P)|.$$

The lemma is proved. ■

Lemma 3.8.4 *With the above notations, let N be a $\mathbf{Q}W_n[\gamma]$-module affording $r\varphi^{\lambda,\emptyset}$ when $\lambda \in DP^+$ and $r(\varphi^{\lambda,\emptyset} + \varepsilon\varphi^{\lambda,\emptyset})$ when $\lambda \in DP^-$, where r is a rational integer, and $W = N \otimes M = N \otimes O \otimes P$, then*

$$dim_{\mathbf{Q}} A_0(W) = dim_{\mathbf{Q}} A_0(N) \times dim_{\mathbf{Q}} A_0(M) \times |L(N) \cap L(M)|,$$

and

$$dim_{\mathbf{Q}} A_0(M) = dim_{\mathbf{Q}} A_0(O) \times dim_{\mathbf{Q}} A_0(P) \times |L(O) \cap L(P)|.$$

Proof: We divide the proof into two cases.

Assume first $\lambda \in DP^-$.

Let N be a $\mathbf{Q}W_n[\gamma]$-module affording $r(\varphi^{\lambda,\emptyset} + \varepsilon\varphi^{\lambda,\emptyset})$.

If $2 \not| n$, we have

$$\Im = 2(\psi + \varepsilon\psi + \delta\psi + \varepsilon\delta\psi), \;\; \psi = \psi^{\varepsilon_1, \varepsilon_2}.$$

It follows that W affords

$$\theta_0 \Im r(\varphi + \varepsilon\varphi) = 16r(\theta_0\psi\varphi), \varphi = \varphi^{\lambda,\emptyset}.$$

Notice that $\psi\varphi$ is irreducible (9.3 of Stembridge [23]), then by Proposition 3.3.4 and Lemma 3.8.2, we get

$$dim_{\mathbf{Q}} A_G(W) \;\; = \;\; ||16r(\theta_0\psi\varphi)||^2$$

$$\begin{aligned} &= 16^2r^2\|(\theta_0\psi\varphi)\|^2 \\ &= 16^2r^2|L(\psi\varphi)| \\ &= 16^2r^2 \\ &= 4^4r^2. \end{aligned}$$

If next $2|n$, we have

$$\Im = 4(\psi + \varepsilon\psi).$$

It follows

$$\theta_0\Im r(\varphi + \varepsilon\varphi) = 16r(\theta_0\psi\varphi).$$

By Proposition 3.3.4 and Lemma 3.8.2, we get

$$\begin{aligned} dim_Q A_G(W) &= \|16r(\theta_0\psi\varphi)\|^2 \\ &= 16^2r^2\|(\theta_0\psi\varphi)\|^2 \\ &= 16^2r^2|L(\psi\varphi)| \\ &= 16^2r^22 \\ &= 4^4r^22. \end{aligned}$$

Similarly, M affords $\theta_0\Im = 8\theta_0\psi$, by Proposition 3.3.4 we have

$$\begin{aligned} dim_{\mathbf{Q}} A_G(M) &= \|8\theta_0\psi\|^2 \\ &= 8^2\|\theta_0\psi\|^2 \\ &= 8^2|L(\psi)| \\ &= \begin{cases} 8^2 & \text{if } 2 \not| n \\ 8^2 \times 2 & \text{if } 2|n. \end{cases} \end{aligned}$$

The last equality follows from Corollary 3.6.6.

Notice $dim_{\mathbf{Q}}A_0(O) = 4$ since $A_0(O) \simeq (1,1)$,

$$dim_{\mathbf{Q}}A_0(N) = \begin{cases} r^2 & \text{if } \lambda \in DP^+ \\ r^2 \times 2 & \text{if } \lambda \in DP^- \end{cases}$$

and

$$dim_{\mathbf{Q}}A_0(P) = \begin{cases} 4^2 & \text{if } 2 \not| n \\ 4^2 \times 2 & \text{if } 2|n. \end{cases}$$

Also notice $L(N) = \{1, \varepsilon\}$, as well as $L(M) = L(O) \cap L(P) = \{1, \varepsilon, \delta, \varepsilon\delta\}$.

It then is straightforward to check that

$$dim_{\mathbf{Q}}A_0(W) = dim_{\mathbf{Q}}A_0(N) \times dim_{\mathbf{Q}}A_0(M) \times |L(N) \cap L(M)|,$$

and

$$dim_{\mathbf{Q}}A_0(M) = dim_{\mathbf{Q}}A_0(O) \times dim_{\mathbf{Q}}A_0(P) \times |L(O) \cap L(P)|.$$

Next assume $\lambda \in DP^+$.

Let N be a $\mathbf{Q}W_n[\gamma]$-module affording $r\varphi^{\lambda,\emptyset}$.

If $2 \not| n$, then in this case

$$\Im = 2(\psi + \varepsilon\psi + \delta\psi + \varepsilon\delta\psi) \text{ where } \psi = \psi^{\varepsilon_1,\varepsilon_2}.$$

Then W affords

$$\theta_0 \Im r\varphi^{\lambda,\emptyset} = 2r\theta_0\varphi^{\lambda,\emptyset}(\psi + \varepsilon\psi + \delta\psi + \varepsilon\delta\psi) = 8r(\theta_0\psi\varphi^{\lambda,\emptyset}).$$

It then follows from Proposition 3.3.4 and Lemma 3.8.2 that

$$\begin{aligned} dim_{\mathbf{Q}}A_G(W) &= ||8r(\theta_0\psi\varphi^{\lambda,\emptyset})||^2 \\ &= 8^2r^2||\theta_0\psi\varphi^{\lambda,\emptyset}||^2 \\ &= 8^2r^2|L(\psi\varphi^{\lambda,\emptyset})| \\ &= 8^2r^2 2 \\ &= 4^3r^2 2. \end{aligned}$$

If next $2|n$, we have

$$\Im = 4(\psi + \varepsilon\psi)$$

and W affords

$$\theta_0 \Im r \varphi^{\lambda,\emptyset} = 4r\theta_0(\psi + \varepsilon\psi)\varphi^{\lambda,\emptyset} = 8r(\theta_0\psi\varphi^{\lambda,\emptyset}).$$

It then follows from Proposition 3.3.4 and Lemma 3.8.2 that

$$\begin{aligned}
dim_{\mathbf{Q}} A_G(W) &= ||8r(\theta_0\psi\varphi^{\lambda,\emptyset})||^2 \\
&= 8^2 r^2 ||(\theta_0\psi\varphi^{\lambda,\emptyset})||^2 \\
&= 8^2 r^2 |L(\psi\varphi^{\lambda,\emptyset})| \\
&= 8^2 r^2 4 \\
&= 4^4 r^2.
\end{aligned}$$

Similarly, M affords $\theta_0 \Im = 8\theta_0\psi$, by Proposition 3.3.4 we have

$$\begin{aligned}
dim_{\mathbf{Q}} A_G(M) &= ||8\theta_0\psi||^2 \\
&= 8^2 ||\theta_0\psi||^2 \\
&= 8^2 |L(\psi)| \\
&= \begin{cases} 8^2 & \text{if } 2 \not| n \\ 8^2 \times 2 & \text{if } 2|n. \end{cases}
\end{aligned}$$

The last equality follows from Corollary 3.6.6.

Notice $dim_{\mathbf{Q}} A_0(O) = 4$ since $A_0(O) \simeq (1,1)$,

$$dim_{\mathbf{Q}} A_0(N) = \begin{cases} r^2 & \text{if } \lambda \in DP^+ \\ r^2 \times 2 & \text{if } \lambda \in DP^- \end{cases}$$

and

$$dim_{\mathbf{Q}} A_0(P) = \begin{cases} 4^2 & \text{if } 2 \not| n \\ 4^2 \times 2 & \text{if } 2|n. \end{cases}$$

Also notice $L(N) = \{1, \varepsilon\}$, as well as $L(M) = L(O) \cap L(P) = \{1, \varepsilon, \delta, \varepsilon\delta\}$.

It then is straightforward to check that

$$dim_{\mathbf{Q}} A_0(W) = dim_{\mathbf{Q}} A_0(N) \times dim_{\mathbf{Q}} A_0(M) \times |L(N) \cap L(M)|,$$

and

$$dim_{\mathbf{Q}} A_0(M) = dim_{\mathbf{Q}} A_0(O) \times dim_{\mathbf{Q}} A_0(P) \times |L(O) \cap L(P)|.$$

The lemma is proved. ■

Because of the above two lemmas, when we apply the theorems of Section 3.7 we don't need to worry about checking the dimensions.

Definition 3.8.5 *Recall the definitions of $S_P^\varepsilon, S_P^\delta$ and J_P from Theorem 3.6.13. Define*

$$\begin{aligned} S_M^\varepsilon &= 1 \otimes S_P^\varepsilon \\ S_M^\delta &= 1 \otimes S_P^\delta \\ J_M &= 1 \otimes J_P. \end{aligned}$$

Remarks : $S_M^\varepsilon, S_M^\delta$ and J_M have the following properties:

1. S_M^ε is an ε-associator of M and $(S_M^\varepsilon)^2 = (S_P^\varepsilon)^2 = 1$; S_M^δ is a δ-associator of M and $(S_M^\delta)^2 = (S_P^\delta)^2 = 1$. Furthermore

$$S_M^\varepsilon S_M^\delta = S_M^\delta S_M^\varepsilon.$$

2. $J_M \in Z(A_0(M)), J_M^2 = J_P^2 = 2\varepsilon_1\varepsilon_2 \in \mathbf{Q}^\times$, and

$$\begin{aligned} J_M S_M^\varepsilon &= -S_M^\varepsilon J_M \\ J_M S_M^\delta &= -S_M^\delta J_M. \end{aligned}$$

In the rest of this section, we will keep the notations $S_P^\varepsilon, S_P^\delta$, and J_P, as well as $S_M^\varepsilon, S_M^\delta$, and J_M without further mention unless otherwise stated.

Theorem 3.8.6 *The structure of $A_0(M)$ can be described as follows.*

If $2 \not| n$, then

$$A_0(O \otimes P) \simeq (-1,-1)^{\binom{n+1}{4}} \otimes (1,1)^2 \otimes \left(-\varepsilon_2(-1)^{\binom{n+1}{2}}, -2\varepsilon_1\varepsilon_2\right).$$

If $2|n$, then

$$A_0(O \otimes P) \simeq (-1,-1)^{\binom{n+1}{4}} \otimes (1,1)^2 \otimes (\varepsilon_2, 2\varepsilon_1) \otimes \mathbf{Q}\left(\sqrt{-2\varepsilon_1\varepsilon_2(-1)^{n/2}}\right).$$

Proof: Since $L(O) \cap L(P) = \{1, \varepsilon, \delta, \varepsilon\delta\}$, by Case 3 of Theorem 3.7.7 we get

$$A_0(O \otimes P) \simeq (1,1) \otimes \Delta,$$

where $\Delta =< 1 \otimes A_0(P), y_0 \otimes S_P^{\varepsilon}, x_0 \otimes S_P^{\delta} >$.

If $2 \not| n$, by Theorem 3.6.13 we have

$$\Delta \simeq (-1,-1)^{\binom{n+1}{4}} \otimes R$$

where

$$\begin{aligned}
R &\simeq \left\langle \begin{array}{l} 1 \otimes \zeta_1 \otimes 1 \otimes 1 \\ 1 \otimes 1 \otimes 1 \otimes a \\ y_0 \otimes \zeta_2 \otimes 1 \otimes b \\ x_0 \otimes 1 \otimes 1 \otimes b \end{array} \right\rangle \\
&\simeq \left\langle \begin{array}{l} \alpha = y_0 \otimes \zeta_2 \otimes 1 \otimes b \\ \beta = x_0 \otimes 1 \otimes 1 \otimes b \\ \gamma = y_0 \otimes \zeta_1\zeta_2 \otimes 1 \otimes ab \\ \omega = x_0 \otimes \zeta_1 \otimes 1 \otimes b \end{array} \right\rangle \\
&\simeq (\alpha^2, \beta^2) \otimes (\gamma^2, \omega^2) \\
&\simeq (1,1) \otimes (\zeta_1^2 a^2, \zeta_1^2) \\
&\simeq (1,1) \otimes \left(-4\varepsilon_2(-1)^{\binom{n+1}{2}}, -2\varepsilon_1(-1)^{\binom{n+1}{2}}\right) \\
&\simeq (1,1) \otimes \left(-\varepsilon_2(-1)^{\binom{n+1}{2}}, -2\varepsilon_1\varepsilon_2\right).
\end{aligned}$$

It follows then

$$A_0(O \otimes P) \simeq (-1,-1)^{\binom{n+1}{4}} \otimes (1,1)^2 \otimes \left(-\varepsilon_2(-1)^{\binom{n+1}{2}}, -2\varepsilon_1\varepsilon_2\right).$$

If $2|n$, by Theorem 3.6.13 we have

$$\Delta \simeq (-1,-1)^{\binom{n+1}{4}} \otimes R$$

where

$$R \simeq \left\langle \begin{array}{l} 1 \otimes \zeta_1 \otimes 1 \otimes 1 \\ 1 \otimes 1 \otimes 1 \otimes a \\ 1 \otimes \zeta_2 \otimes \xi \otimes 1 \\ y_0 \otimes \zeta_2 \otimes 1 \otimes b \\ x_0 \otimes 1 \otimes 1 \otimes b \end{array} \right\rangle$$

$$\simeq \left\langle \begin{array}{l} \alpha = y_0 \otimes \zeta_2 \otimes 1 \otimes b \\ \beta = x_0 \otimes 1 \otimes 1 \otimes b \\ \gamma = y_0 \otimes \zeta_1\zeta_2 \otimes 1 \otimes ab \\ \omega = x_0 \otimes \zeta_1 \otimes 1 \otimes b \\ \eta = x_0y_0 \otimes 1 \otimes \xi \otimes a \end{array} \right\rangle$$

$$\begin{aligned}
&\simeq (\alpha^2, \beta^2) \otimes (\gamma^2, \omega^2) \otimes <\eta> \\
&\simeq (1,1) \otimes (\zeta_1^2 a^2, \zeta_1^2) \otimes \mathbf{Q}(\sqrt{\eta^2}) \\
&\simeq (1,1) \otimes \left(-4\varepsilon_2(-1)^{\binom{n+1}{2}}, -2\varepsilon_1(-1)^{\binom{n+1}{2}}\right) \otimes \mathbf{Q}\left(\sqrt{-2\varepsilon_1\varepsilon_2(-1)^{n/2}}\right) \\
&\simeq (1,1) \otimes \left(-\varepsilon_2(-1)^{\binom{n+1}{2}}, -2\varepsilon_1\varepsilon_2\right) \otimes \mathbf{Q}\left(\sqrt{-2\varepsilon_1\varepsilon_2(-1)^{n/2}}\right).
\end{aligned}$$

Notice $(-1)^{\binom{n+1}{2}} = (-1)^{n/2}$ when $2|n$. Then it follows from Proposition 2.5.1 that

$$A_0(O \otimes P) \simeq (-1,-1)^{\binom{n+1}{4}} \otimes (1,1)^2 \otimes (\varepsilon_2, 2\varepsilon_1) \otimes \mathbf{Q}\left(\sqrt{-2\varepsilon_1\varepsilon_2(-1)^{n/2}}\right).$$

The proof of the theorem is complete. ∎

We finish this section by proving some other lemmas which will are needed in Section 4.6 and 4.8 to simplify the calculation of algebras related with triple product.

Lemma 3.8.7 *The structure of the centralizer of S_P^ε in $A_0(P)$ can be described as follows.*

If $2 \not| n$, then

$$\begin{aligned}
C_{A_0(P)}(S_P^\varepsilon) &\simeq (-1,-1)^{\binom{n+1}{4}} \otimes <1 \otimes \zeta_1 \otimes 1 \otimes a> \\
&\simeq (-1,-1)^{\binom{n+1}{4}} \otimes \mathbf{Q}\left(\sqrt{-\varepsilon_2(-1)^{\binom{n+1}{2}}}\right).
\end{aligned}$$

If $2|n$, then

$$\begin{aligned}
C_{A_0(P)}(S_P^\varepsilon) &\simeq (-1,-1)^{\binom{n+1}{4}} \otimes \left\langle \begin{array}{c} 1 \otimes \zeta_1 \otimes 1 \otimes a \\ 1 \otimes \zeta_2 \otimes \xi \otimes 1 \end{array} \right\rangle \\
&\simeq (-1,-1)^{\binom{n+1}{4}} \otimes (\varepsilon_2, (-1)^{n/2}).
\end{aligned}$$

Proof: Notice that $(-1)^{\binom{n+1}{2}} = (-1)^{n/2}$ when $2|n$. Then result follows straightforward from the definition of S_P^ε and Theorem 3.6.13. ∎

Lemma 3.8.8 *The structure of The centralizer of S_M^ε in $A_0(M)$ can be described as follows.*

If $2 \not| n$, then

$$C_{A_0(M)}(S_M^\varepsilon) \simeq (-1,-1)^{\binom{n+1}{4}} \otimes (1,1)^2 \otimes \mathbf{Q}\left(\sqrt{-\varepsilon_2(-1)^{\binom{n+1}{2}}}\right).$$

If $2|n$, then

$$C_{A_0(M)}(S_M^\varepsilon) \simeq (-1,-1)^{\binom{n+1}{4}} \otimes (1,1)^2 \otimes \left(\varepsilon_2, (-1)^{n/2}\right).$$

Proof: It follows from the dimension count and the definition of S_M^ε that

$$C_{A_0(M)}(S_M^\varepsilon) \quad \simeq \quad A_0(O) \otimes R$$

where

$$R = \left\langle \begin{array}{l} 1 \otimes C_{A_0(P)}(S_P^\varepsilon) \\ y_0 \otimes S_P^\varepsilon \\ x_0 \otimes S_P^\delta \end{array} \right\rangle.$$

If $2 \not| n$, by Lemma 3.8.7 we get

$$\begin{aligned}
R &\simeq (-1,-1)^{\binom{n+1}{4}} \otimes \left\langle \begin{array}{l} 1 \otimes \zeta_1 \otimes a \\ y_0 \otimes \zeta_2 \otimes b \\ x_0 \otimes 1 \otimes b \end{array} \right\rangle \\
&\simeq (-1,-1)^{\binom{n+1}{4}} \otimes \left\langle \begin{array}{l} \alpha = 1 \otimes \zeta_1 \otimes a \\ \beta = x_0 \otimes 1 \otimes b \\ \gamma = y_0 \otimes \zeta_1\zeta_2 \otimes ab \end{array} \right\rangle \\
&\simeq (-1,-1)^{\binom{n+1}{4}} \otimes < \alpha, \beta > \otimes < \gamma > \\
&\simeq (-1,-1)^{\binom{n+1}{4}} \otimes (\alpha^2, \beta^2) \otimes \mathbf{Q}\left(\sqrt{\gamma^2}\right) \\
&\simeq (-1,-1)^{\binom{n+1}{4}} \otimes (-4\varepsilon_2(-1)^{\binom{n+1}{2}}, 1) \otimes \mathbf{Q}\left(\sqrt{-4\varepsilon_2(-1)^{\binom{n+1}{2}}}\right) \\
&\simeq (-1,-1)^{\binom{n+1}{4}} \otimes (1,1) \otimes \mathbf{Q}\left(\sqrt{-\varepsilon_2(-1)^{\binom{n+1}{2}}}\right).
\end{aligned}$$

It follows then

$$C_{A_0(M)}(S_M) \simeq (-1,-1)^{\binom{n+1}{4}} \otimes (1,1)^2 \otimes \mathbf{Q}\left(\sqrt{-\varepsilon_2(-1)^{\binom{n+1}{2}}}\right).$$

If $2|n$, by Lemma 3.8.7 we get

$$\begin{aligned}
R &\simeq (-1,-1)^{\binom{n+1}{4}} \otimes \left\langle \begin{array}{l} 1\otimes\zeta_1\otimes 1\otimes a \\ 1\otimes\zeta_2\otimes\xi\otimes 1 \\ y_0\otimes\zeta_2\otimes 1\otimes b \\ x_0\otimes 1\otimes 1\otimes b \end{array} \right\rangle \\
&\simeq (-1,-1)^{\binom{n+1}{4}} \otimes \left\langle \begin{array}{l} \alpha = y_0\otimes\zeta_2\otimes 1\otimes b \\ \beta = x_0\otimes\zeta_2\otimes\xi\otimes b \\ \gamma = 1\otimes\zeta_2\otimes\xi\otimes 1 \\ \omega = 1\otimes\zeta_1\otimes 1\otimes a \end{array} \right\rangle \\
&\simeq (-1,-1)^{\binom{n+1}{4}} \otimes <\alpha,\beta> \otimes <\gamma,\omega> \\
&\simeq (-1,-1)^{\binom{n+1}{4}} \otimes (\alpha^2,\beta^2) \otimes (\gamma^2,\omega^2) \\
&\simeq (-1,-1)^{\binom{n+1}{4}} \otimes (1,\xi^2) \otimes (\xi^2,\zeta_1^2 a^2) \\
&\simeq (-1,-1)^{\binom{n+1}{4}} \otimes (1,1) \otimes \left((-1)^{n/2}, -\varepsilon_2(-1)^{\binom{n+1}{2}}\right).
\end{aligned}$$

Notice $(-1)^{\binom{n+1}{2}} = (-1)^{n/2}$ when $2|n$. It then follows from Proposition 2.5.1 that

$$C_{A_0(M)}(S_M) \simeq (-1,-1)^{\binom{n+1}{4}} \otimes (1,1)^2 \otimes \left(\varepsilon_2, (-1)^{n/2}\right).$$

The lemma is proved. ∎

Lemma 3.8.9 *Let N, O, P be as above and Let $M = O \otimes P$ be the twisted product of O and P. Assume $L(N) = \{1, \varepsilon\}$, let H be the kernel of ε, and consider the group pair $(W_n[\beta], H)$. Assume the graded algebra $A = A_H(N)$ is a* **CSGA** *of even type and $dim_{\mathbf{Q}} A = 2 \times dim_{\mathbf{Q}} A_0$. Let*

$$\begin{aligned}
A &\simeq [D,0,d], A_0 = A_0(N) \\
Z(A_0) &\simeq \mathbf{Q} + i\mathbf{Q}, i^2 = d \in \mathbf{Q}^\times
\end{aligned}$$

where D can be chosen to be a quaternion algebra.

Let

$$\Delta = \left\langle 1 \otimes C_{A_0(M)}(S_M^\varepsilon), i \otimes J_M \right\rangle.$$

Then the structure of Δ can be described as follows.

If $2\not|n$, then

$$\Delta \simeq (-1,-1)^{\binom{n+1}{4}} \otimes (1,1)^2 \otimes \left(-\varepsilon_2(-1)^{\binom{n+1}{2}}, -2\varepsilon_1\varepsilon_2 d\right).$$

If $2|n$, then

$$\Delta \simeq (-1,-1)^{\binom{n+1}{4}} \otimes (1,1)^2 \otimes \left(\varepsilon_2, (-1)^{n/2}\right) \otimes \mathbf{Q}\left(\sqrt{-2\varepsilon_1\varepsilon_2 d(-1)^{n/2}}\right).$$

Proof: By the dimension count and the definition of S_M^ε we get

$$\begin{aligned}\Delta &= A_0(O) \otimes \left\langle \begin{array}{l} 1\otimes 1\otimes C_{A_0(P)}(S_P^\varepsilon) \\ 1\otimes y_0\otimes S_P^\varepsilon \\ 1\otimes x_0 \otimes S_P^\delta \\ i\otimes 1\otimes J_P \end{array}\right\rangle \\ &= (1,1)\otimes \left\langle \begin{array}{l} 1\otimes 1\otimes C_{A_0(P)}(S_P^\varepsilon) \\ 1\otimes y_0\otimes S_P^\varepsilon \\ 1\otimes x_0 \otimes S_P^\delta \\ i\otimes 1\otimes J_P \end{array}\right\rangle.\end{aligned}$$

If $2\not|n$, then

$$\begin{aligned}\Delta &\simeq (-1,-1)^{\binom{n+1}{4}} \otimes (1,1)\otimes \left\langle \begin{array}{l} 1\otimes 1\otimes 1\otimes \zeta_1\otimes 1\otimes a \\ 1\otimes y_0\otimes 1\otimes \zeta_2\otimes 1\otimes b \\ 1\otimes x_0\otimes 1\otimes 1\otimes 1\otimes b \\ i\otimes 1\otimes 1\otimes 1\otimes 1\otimes a \end{array}\right\rangle \\
&\simeq (-1,-1)^{\binom{n+1}{4}} \otimes (1,1)\otimes \left\langle \begin{array}{l} 1\otimes 1\otimes \zeta_1\otimes a \\ 1\otimes y_0\otimes \zeta_2\otimes b \\ 1\otimes x_0\otimes 1\otimes b \\ i\otimes 1\otimes 1\otimes a \end{array}\right\rangle \\
&\simeq (-1,-1)^{\binom{n+1}{4}} \otimes (1,1)\left\langle \begin{array}{l} \alpha = 1\otimes y_0\otimes \zeta_2\otimes b \\ \beta = i\otimes 1\otimes 1\otimes a \\ \gamma = 1\otimes 1\otimes \zeta_1\otimes a \\ \omega = i\otimes x_0y_0\otimes \zeta_1\zeta_2\otimes 1 \end{array}\right\rangle \\
&\simeq (-1,-1)^{\binom{n+1}{4}} \otimes (1,1)\otimes <\alpha,\beta> \otimes <\gamma,\omega> \\
&\simeq (-1,-1)^{\binom{n+1}{4}} \otimes (1,1)\otimes (\alpha^2,\beta^2)\otimes(\gamma^2,\omega^2) \\
&\simeq (-1,-1)^{\binom{n+1}{4}} \otimes (1,1)\otimes (y_0^2\zeta_2^2b^2, i^2a^2)\otimes(\zeta_1^2a^2, i^2(x_0y_0)^2(\zeta_1\zeta_2)^2) \\
&\simeq (-1,-1)^{\binom{n+1}{4}} \otimes (1,1)\otimes (1, i^2a^2)\otimes\left(-4\varepsilon_2(-1)^{\binom{n+1}{2}}, -2d\varepsilon_1(-1)^{\binom{n+1}{2}}\right).\end{aligned}$$

It then follows from Proposition 2.5.1

$$\Delta \simeq (-1,-1)^{\binom{n+1}{4}} \otimes (1,1)^2 \otimes \left(-\varepsilon_2(-1)^{\binom{n+1}{2}}, -2d\varepsilon_1\varepsilon_2\right).$$

If $2|n$, then

$$\begin{aligned}
\Delta &\simeq (-1,-1)^{\binom{n+1}{4}} \otimes (1,1) \otimes \left\langle \begin{array}{l} 1\otimes 1\otimes 1\otimes \zeta_1\otimes 1\otimes a \\ 1\otimes 1\otimes 1\otimes \zeta_2\otimes \xi\otimes 1 \\ 1\otimes y_0\otimes 1\otimes \zeta_2\otimes 1\otimes b \\ 1\otimes x_0\otimes 1\otimes 1\otimes 1\otimes b \\ i\otimes 1\otimes 1\otimes 1\otimes 1\otimes a \end{array} \right\rangle \\
&\simeq (-1,-1)^{\binom{n+1}{4}} \otimes (1,1) \otimes \left\langle \begin{array}{l} 1\otimes 1\otimes \zeta_1\otimes 1\otimes a \\ 1\otimes 1\otimes \zeta_2\otimes \xi\otimes 1 \\ 1\otimes y_0\otimes \zeta_2\otimes 1\otimes b \\ 1\otimes x_0\otimes 1\otimes 1\otimes b \\ i\otimes 1\otimes 1\otimes 1\otimes a \end{array} \right\rangle \\
&\simeq (-1,-1)^{\binom{n+1}{4}} \otimes (1,1) \otimes \left\langle \begin{array}{l} \alpha = 1\otimes y_0\otimes \zeta_2\otimes 1\otimes b \\ \beta = i\otimes 1\otimes 1\otimes 1\otimes a \\ \gamma = 1\otimes 1\otimes \zeta_1\otimes 1\otimes a \\ \omega = i\otimes x_0y_0\otimes \zeta_1\zeta_2\otimes 1\otimes 1 \\ \eta = i\otimes x_0y_0\otimes 1\otimes \xi\otimes a \end{array} \right\rangle \\
&\simeq (-1,-1)^{\binom{n+1}{4}} \otimes (1,1) \otimes <\alpha,\beta> \otimes <\gamma,\omega> \otimes <\eta> \\
&\simeq (-1,-1)^{\binom{n+1}{4}} \otimes (1,1) \otimes (\alpha^2,\beta^2) \otimes (\gamma^2,\omega^2) \otimes \mathbf{Q}(\sqrt{\eta^2}) \\
&\simeq (-1,-1)^{\binom{n+1}{4}} \otimes (1,1) \otimes (y_0^2\zeta_2^2 b^2, i^2a^2) \otimes (\zeta_1^2 a^2, i^2(x_0y_0)^2(\zeta_1\zeta_2)^2) \\
&\qquad \otimes \mathbf{Q}\left(\sqrt{i^2(x_0y_0)^2\xi^2a^2}\right) \\
&\simeq (-1,-1)^{\binom{n+1}{4}} \otimes (1,1) \otimes (1, i^2a^2) \otimes \left(-4\varepsilon_2(-1)^{\binom{n+1}{2}}, -2d\varepsilon_1(-1)^{\binom{n+1}{2}}\right) \\
&\qquad \otimes \mathbf{Q}\left(\sqrt{-2d\varepsilon_1\varepsilon_2(-)^{n/2}}\right)
\end{aligned}$$

It then follows from Proposition 2.5.1 that

$$\Delta \simeq (-1,-1)^{\binom{n+1}{4}} \otimes (1,1)^2 \otimes (\varepsilon_2, (-1)^{n/2}) \otimes \mathbf{Q}\left(\sqrt{-2d\varepsilon_1\varepsilon_2(-1)^{n/2}}\right).$$

The lemma is proved. ■

CHAPTER 4
SCHUR INDICES

In this chapter, we turn to the problem of calculating the division algebras associated with (spin) characters for each of the eight families of double covers of W_n. This task can be roughly divided into two phases. In the first phase, we construct the modules and describe the characters (or spin representations) for each of the eight families of double covers of W_n, which follows Stembridge's idea [23]. In the second phase, we use group pair technique to calculate the division algebras for each (spin) character, which follows Turull's idea [25]. The algebras constructed, which is associated with each irreducible representation, can be used to calculate the Schur index of the spin representation.

4.1 $[\varepsilon_1, \varepsilon_2, +1, -1, +1]$

In this section, we will study the spin characters of double cover $W_n[\varepsilon_1, \varepsilon_2, +1, -1, +1]$ and calculate the division algebra associated with each irreducible spin representation. Let $\Theta^{\varepsilon_1,\varepsilon_2}$ be the spin representation of $W_n[\varepsilon_1, \varepsilon_2, +1, -1, +1]$ constructed in Section 2.4, and $\theta^{\varepsilon_1,\varepsilon_2}$ be the corresponding character. Since $\Theta^{\varepsilon_1,\varepsilon_2}$ is self-dual, according to the program laid out in Section 2.4, the constructions of these representations (if not already irreducible) can be constructed from the associators of $\Theta^{\varepsilon_1,\varepsilon_2}$ and $X^{\lambda,\mu}$ with respect to each of the linear characters of the two groups. As the first result, we have the following theorem:

Theorem 4.1.1 *The irreducible spin representation of $W_n[\varepsilon_1, \varepsilon_2, +1, -1, +1]$ can be labeled by L_n-orbits of pairs of partitions (λ, μ) with $|\lambda| + |\mu| = n$, so that the orbit of (λ, μ) labels submodules of $\Theta^{\varepsilon_1,\varepsilon_2} X^{\lambda,\mu}$. In the following table, $n_{\lambda,\mu}$ denotes the number*

of modules indexed by (λ, μ)*,* $m_{\lambda,\mu}$ *denotes their multiplicity in* $\Theta^{\varepsilon_1,\varepsilon_2}X^{\lambda,\mu}$*, and* $o_{\lambda,\mu}$ *denotes the size of the orbit of* (λ, μ)*.*

$n_{\lambda,\mu}$	$o_{\lambda,\mu}$	$m_{\lambda,\mu}$	
1	1	4	if $\lambda = \mu \in SC, n \equiv 0 \ mod \ 4$
4	1	2	if $\lambda = \mu \in SC, n \equiv 2 \ mod \ 4$
2	2	2	if $\lambda, \mu \in SC,$ or $\lambda = \mu$ or $\lambda = \mu'$, but not $\lambda = \mu \in SC$
1	4	2	otherwise.

Proof: By Proposition 3.3.3 we have $\Theta^{\varepsilon_1,\varepsilon_2} = 2\Theta_0^{\varepsilon_1,\varepsilon_2}$. Then the case $\varepsilon_1 = 1 = \varepsilon_2$ follows readily from Corollary 6.5 of Stembridge [23]. The remaining cases follow from the Stembridge's argument, using Theorem 3.3.5 and Theorem 2.4.3. ■

The next theorem characterizes the central simple algebra associated with each irreducible spin representation indexed by a pair of partitions (λ, μ) with $|\lambda| + |\mu| = n$ as defined in above theorem.

Theorem 4.1.2 *Let* $\alpha = [\varepsilon_1, \varepsilon_2, +1, -1, +1]$*,* $\chi^{\lambda,\mu}$ *be any character indexed by* λ *and* μ *in above theorem, and* $A^{\lambda,\mu}(\alpha)$*, as an element of* $Br\left(\mathbf{Q}(\chi^{\lambda,\mu})\right)$*, be the central simple algebra associated with* $\chi^{\lambda,\mu}$*. Recall the definition of* Z_λ^* *in Definition 3.1.4. Then we have the following:*

(1) If $\lambda, \mu \in SC, \lambda \neq \mu$*, then*

$$A^{\lambda,\mu}(\alpha) = (\varepsilon_1, \varepsilon_2),$$

with $\mathbf{Q}(\chi^{\lambda,\mu}) = \mathbf{Q}\left(\sqrt{\varepsilon_2 Z_\lambda^* Z_\mu^*}\right)$.

(2) If $\lambda = \mu'$*, but not both of* $\lambda, \mu \in SC$*, then*

$$A^{\lambda,\mu}(\alpha) = (\varepsilon_1, \varepsilon_2),$$

with $\mathbf{Q}(\chi^{\lambda,\mu}) = \mathbf{Q}\left(\sqrt{\varepsilon_1\varepsilon_2(-1)^{\frac{n}{2}+1}}\right)$.

(3) If $\lambda = \mu$*, but not both of* $\lambda, \mu \in SC$*, then*

$$A^{\lambda,\mu}(\alpha) = 1,$$

with $\mathbf{Q}(\chi^{\lambda,\mu}) = \mathbf{Q}(\sqrt{\varepsilon_1})$.

(4) If $\lambda = \mu \in SC$ and $n \equiv 0$ (mod 4), then

$$A^{\lambda,\mu}(\alpha) = 1,$$

with $\mathbf{Q}(\chi^{\lambda,\mu}) = \mathbf{Q}$.

(5) If $\lambda = \mu \in SC$ and $n \equiv 2$ (mod 4), then

$$A^{\lambda,\mu}(\alpha) = 1,$$

with $\mathbf{Q}(\chi^{\lambda,\mu}) = \mathbf{Q}\left(\sqrt{\varepsilon_1}, \sqrt{\varepsilon_2}\right)$.

(6) For all other cases,

$$A^{\lambda,\mu}(\alpha) = (\varepsilon_1, \varepsilon_2),$$

with $\mathbf{Q}(\chi^{\lambda,\mu}) = \mathbf{Q}$.

Proof: Let $G = W_n[\varepsilon_1, \varepsilon_2, +1, -1, +1]$. Let N be a $\mathbf{Q}W_n[+1, +1, +1, +1, +1]$-module affording $X^{\lambda,\mu}$, and let O be a $\mathbf{Q}W_n[\varepsilon_1, \varepsilon_2, +1, -1, +1]$-module affording $\Theta^{\varepsilon_1,\varepsilon_2}$, then $W = O \otimes N$, as a $\mathbf{Q}G$-module, affords the twisted product $\Theta^{\varepsilon_1,\varepsilon_2}X^{\lambda,\mu}$. We want to compute $A_0(W) := End_{\mathbf{Q}G}(W)$. As is known, $A_0(W)$ is a semi-simple algebra over $\mathbf{Q}$, and $A^{\lambda,\mu}(\alpha)$ is one of its direct summands.

By Proposition 3.3.4, we have

$$||\theta^{\varepsilon_1,\varepsilon_2}\chi^{\lambda,\mu}||^2 = 4|L(\chi^{\lambda,\mu})|.$$

Notice $A_0(O) \simeq (\varepsilon_1, \varepsilon_2)$, $A_0(N) \simeq \mathbf{Q}$, and $L(O) = \{1, \varepsilon, \delta, \varepsilon\delta\}$. We have

$$\begin{aligned} dim_{\mathbf{Q}} A_0(W) &= ||\theta^{\varepsilon_1,\varepsilon_2}\chi^{\lambda,\mu}||^2 \\ &= 4|L(\chi^{\lambda,\mu})| \\ &= dim_{\mathbf{Q}} A_0(O) \times dim_{\mathbf{Q}} A_0(N) \times |L(O) \cap L(N)|. \end{aligned}$$

If $|L(N)| = 1$, by Theorem 3.7.7, we have

$$\begin{aligned} A_0(W) &\simeq A_G(O) \otimes A_0(N) \\ &\simeq (\varepsilon_1, \varepsilon_2) \otimes \mathbf{Q} \\ &\simeq (\varepsilon_1, \varepsilon_2). \end{aligned}$$

It then follows from Theorem 2.6.2 that

$$A^{\lambda,\mu}(\alpha) = (\varepsilon_1, \varepsilon_2)$$

in $Br(\mathbf{Q}(\chi^{\lambda,\mu}))$ with $\mathbf{Q}(\chi^{\lambda,\mu}) = \mathbf{Q}$.

Then conclusion (6) is proved.

Next consider the case $|L(N)| = 2$. If $L(N) = \{1, \nu\}$, let S_N^{ν} be a ν-associator of N.

Consider the group pair $(W_n[+1,+1,+1,+1,+1], H)$, where H is the kernel of ε. By Corollary 3.5.8, $A := End_{\mathbf{Q}H}(N)$ with $A_0 = A_0(N)$. is a **CSGA** of odd type.

By Theorem 3.7.7, we have

$$\begin{aligned} A_0(W) &\simeq (\varepsilon_1, \varepsilon_2) \otimes A_0(N) \otimes \mathbf{Q}\left(\sqrt{(S_N^{\nu})^2 (S_O^{\nu})^2}\right) \\ &\simeq (\varepsilon_1, \varepsilon_2) \otimes \mathbf{Q} \otimes \mathbf{Q}\left(\sqrt{(S_N^{\nu})^2 (S_O^{\nu})^2}\right). \end{aligned}$$

If $\nu = \varepsilon$, then $(S_N^{\varepsilon})^2 = Z_\lambda^* Z_\mu^*$ and $(S_O^{\varepsilon})^2 = y_0^2 = \varepsilon_2$. It then follows from Theorem 2.6.2 that

$$A^{\lambda,\mu}(\alpha) = (\varepsilon_1, \varepsilon_2)$$

in $Br\left(\mathbf{Q}(\chi^{\lambda,\mu})\right)$ with $\mathbf{Q}\left(\chi^{\lambda,\mu}\right) = \mathbf{Q}\left(\sqrt{\varepsilon_2 Z_\lambda^* Z_\mu^*}\right)$.

Then conclusion (1) is proved.

If $\nu = \delta$, then $(S_N^{\delta})^2 = 1$ and $(S_O^{\varepsilon})^2 = x_0^2 = \varepsilon_1$. It then follows from Theorem 2.6.2 and Proposition 2.5.1 that

$$A^{\lambda,\mu}(\alpha) = (\varepsilon_1, \varepsilon_2) = 1$$

in $Br\left(\mathbf{Q}(\chi^{\lambda,\mu})\right)$ with $\mathbf{Q}(\chi^{\lambda,\mu}) = \mathbf{Q}\left(\sqrt{\varepsilon_1}\right)$.

Then conclusion (3) is proved.

If $\nu = \varepsilon\delta$, then $(S_N^{\varepsilon\delta})^2 = (-1)^{n/2}$ and $(S_O^{\varepsilon\delta})^2 = (x_0 y_0)^2 = -1$. It then follows from Theorem 2.6.2 that

$$A^{\lambda,\mu}(\alpha) = (\varepsilon_1, \varepsilon_2)$$

in $Br\left(\mathbf{Q}(\chi^{\lambda,\mu})\right)$ with $\mathbf{Q}(\chi^{\lambda,\mu}) = \mathbf{Q}\left(\sqrt{-\varepsilon_1\varepsilon_2(-1)^{n/2}}\right)$.

Then conclusion (2) is proved.

Finally, suppose $|L(N)| = 4$. In this case, we have $L(N) = L(\chi^{\lambda,\mu}) = \{1, \varepsilon, \delta, \varepsilon\delta\}$ and $\mathbf{Q}(\chi) = \mathbf{Q}$. So we are in the third situation of Theorem 3.7.7.

Let S and T be an ε-associator and a δ-associator respectively. Then $S^2 = Z_\lambda^* Z_\mu^*$ and $T^2 = 1$.

- Case: $\lambda = \mu \in SC$ and $n \equiv 0$ $(mod\ 4)$.

In this case, by Theorem 3.1.7 $ST = TS$. It then follows from Theorem 3.7.7 that

$$\begin{aligned} A_0(W) &\simeq A_0(O) \otimes A_0(M) \otimes (S^2\varepsilon_2, T^2\varepsilon_1) \\ &\simeq (\varepsilon_1, \varepsilon_2) \otimes \mathbf{Q} \otimes (\varepsilon_2 Z_\lambda^* Z_\lambda^*, \varepsilon_1). \end{aligned}$$

It then follows from Theorem 2.6.2 and Proposition 2.5.1

$$A^{\lambda,\mu}(\alpha) = 1$$

in $Br\left(\mathbf{Q}(\chi^{\lambda,\mu})\right)$ with $\mathbf{Q}(\chi^{\lambda,\mu}) = \mathbf{Q}$.

The conclusion (4) is proved.

- Case: $\lambda = \mu \in SC$ and $n \equiv 2$ $(mod\ 4)$.

In this case, by Theorem 3.1.7 $ST = -TS$. It then follows from Theorem 3.7.7 that

$$\begin{aligned} A_0(W) &\simeq A_0(O) \otimes A_0(M) \otimes R \\ &\simeq (\varepsilon_1, \varepsilon_2) \otimes \mathbf{Q} \otimes R, \end{aligned}$$

where R is a direct sum of fields and each of its summands is isomorphic to

$$\mathbf{Q}\left(\sqrt{\varepsilon_2 S^2}, \sqrt{\varepsilon_1 T^2}\right) \simeq \mathbf{Q}\left(\sqrt{\varepsilon_2}, \sqrt{\varepsilon_1}\right).$$

Then Corollary 2.8.5 implies

$$A^{\lambda,\mu}(\alpha) \simeq (\varepsilon_1, \varepsilon_2) \otimes \mathbf{Q}\left(\sqrt{\varepsilon_1}, \sqrt{\varepsilon_2}\right).$$

It then follows from Theorem 2.6.2 and Proposition 2.5.1 that

$$A^{\lambda,\mu}(\alpha) = 1$$

in $Br\left(\mathbf{Q}(\chi^{\lambda,\mu})\right)$ with $\mathbf{Q}(\chi^{\lambda,\mu}) = \mathbf{Q}\left(\sqrt{\varepsilon_1}, \sqrt{\varepsilon_2}\right)$.

The conclusion (5) is proved.

The proof of the theorem is now complete. ∎

Corollary 4.1.3 *Let $\alpha = [\varepsilon_1, \varepsilon_2, +1, -1, +1]$, $A^{\lambda,\mu}(\alpha)$ be the central simple algebra associated with the character $\chi^{\lambda,\mu}$ indexed by λ and μ in the above theorem, then $A^{\lambda,\mu}(\alpha) = (\varepsilon_1, \varepsilon_2)$ in $Br\left(\mathbf{Q}(\chi^{\lambda,\mu})\right)$, except the case $A^{\lambda,\mu}(\alpha) = 1$ in $Br\left(\mathbf{Q}(\chi^{\lambda,\mu})\right) = Br\left(\mathbf{Q}\right)$ when $\lambda = \mu \in SC$ and $n \equiv 0$ (mod 4).*

Proof: This follows from above theorem and Theorem 2.6.2. ∎

4.2 $[\varepsilon_1, \varepsilon_2, +1, +1, +1]$

In this section, we will study the spin characters of the double cover $W_n[\varepsilon_1, \varepsilon_2, +1, +1, +1]$ and calculate the division algebra associated with each irreducible spin representation. Let $\rho = \rho^{\varepsilon_1,\varepsilon_2}$ be the sum of the four linear spin representations of $W_n[\varepsilon_1, \varepsilon_2, +1, +1, +1]$, then ρ is self-dual. According to Lemma 2.1.6, the spin representations of $W_n[\varepsilon_1, \varepsilon_2, +1, +1, +1]$ come from the constituents of the twisted product $\rho X^{\lambda,\mu}$. We have the following theorem:

Theorem 4.2.1 *The irreducible spin representation of $W_n[\varepsilon_1, \varepsilon_2, +1, +1, +1]$ can be labeled by L_n-orbits of pairs of partitions (λ, μ) with $|\lambda| + |\mu| = n$, so that the orbit of (λ, μ) labels submodules of $\rho X^{\lambda,\mu}$. In the following table, $n_{\lambda,\mu}$ denotes the number of modules indexed by (λ, μ), $m_{\lambda,\mu}$ denotes their multiplicity in $\rho X^{\lambda,\mu}$, and $o_{\lambda,\mu}$ denotes the size of the orbit of (λ, μ).*

$n_{\lambda,\mu}$	$o_{\lambda,\mu}$	$m_{\lambda,\mu}$	
1	1	4	if $\lambda = \mu \in SC$
2	2	2	if $\lambda, \mu \in SC$, or $\lambda = \mu$, or $\lambda = \mu'$, but not $\lambda = \mu \in SC$
4	4	1	otherwise.

Proof: The theorem follows from Lemma 2.1.6 and Theorem 2.4.3. ∎

Theorem 4.2.2 *Let* $\alpha = [\varepsilon_1, \varepsilon_2, +1, +1, +1]$, $\chi^{\lambda,\mu}$ *be any character indexed by the above theorem, and let* $A^{\lambda,\mu}(\alpha)$, *as an element of* $Br(\mathbf{Q}(\chi^{\lambda,\mu}))$, *be the central simple algebra associated with the character* $\chi^{\lambda,\mu}$. *Recall the definition of* Z^*_λ *in Definition 3.1.4. Then we have the following:*

(1) If $\lambda, \mu \in SC$, *and* $\lambda \neq \mu$, *then*

$$A^{\lambda,\mu}(\alpha) = (\varepsilon_1, Z^*_\lambda Z^*_\mu),$$

with $\mathbf{Q}(\chi^{\lambda,\mu}) = \mathbf{Q}(\sqrt{\varepsilon_2})$.

(2) If $\lambda = \mu$, *but not both of* $\lambda, \mu \in SC$, *then*

$$A^{\lambda,\mu}(\alpha) = 1,$$

with $\mathbf{Q}(\chi^{\lambda,\mu}) = \mathbf{Q}(\sqrt{\varepsilon_1})$.

(3) If $\lambda = \mu'$, *but not both of* $\lambda, \mu \in SC$, *then*

$$A^{\lambda,\mu}(\alpha) = (\varepsilon_2, (-1)^{\frac{n}{2}}),$$

with $\mathbf{Q}(\chi^{\lambda,\mu}) = \mathbf{Q}(\sqrt{\varepsilon_1 \varepsilon_2})$.

(4) If $\lambda = \mu \in SC$ *and* $n \equiv 0$ *(mod 4), then*

$$A^{\lambda,\mu}(\alpha) = 1,$$

with $\mathbf{Q}(\chi^{\lambda,\mu}) = \mathbf{Q}$.

(5) If $\lambda = \mu \in SC$ *and* $n \equiv 2$ *(mod 4), then*

$$A^{\lambda,\mu}(\alpha) = (\varepsilon_1, \varepsilon_2),$$

with $\mathbf{Q}(\chi^{\lambda,\mu}) = \mathbf{Q}$.

(6) For all other cases,

$$A^{\lambda,\mu}(\alpha) = 1.$$

with $\mathbf{Q}(\chi^{\lambda,\mu}) = \mathbf{Q}$ *if* $\varepsilon_1 = 1 = \varepsilon_2$, $\mathbf{Q}(\sqrt{-1})$ *otherwise.*

Proof: Let $G = W_n[\varepsilon_1, \varepsilon_2, +1, +1, +1]$. Let M be a $\mathbf{Q}W_n[+1, +1, +1, +1, +1]$-module affording $X^{\lambda,\mu}$, N be a $\mathbf{Q}W_n[\varepsilon_1, \varepsilon_2, +1, +1, +1]$-module affording

$$\rho^{\varepsilon_1,\varepsilon_2} = \Upsilon + \varepsilon\Upsilon + \delta\Upsilon + \varepsilon\delta\Upsilon,$$

the sum of the four linear spin representations of G. Then $W = N \otimes M$, as a $\mathbf{Q}G$-module, affords the twisted product $\rho^{\varepsilon_1,\varepsilon_2} X^{\lambda,\mu}$. Our purpose is to compute $A_0(W) := End_{\mathbf{Q}G}(W)$, which provides us with the information about $A^{\lambda,\mu}(\alpha)$.

By Proposition 3.3.4, we have

$$\|\varrho^{\varepsilon_1,\varepsilon_2}\chi^{\lambda,\mu}\|^2 = 4|L(\chi^{\lambda,\mu})|.$$

Notice $dim_{\mathbf{Q}} A_0(N) = 4$, $A_0(M) \simeq \mathbf{Q}$ and $L(N) = \{1, \varepsilon, \delta, \varepsilon\delta\}$. We have

$$\begin{aligned} dim_{\mathbf{Q}} A_0(W) &= \|\varrho^{\varepsilon_1,\varepsilon_2}\chi^{\lambda,\mu}\|^2 \\ &= 4|L(\chi^{\lambda,\mu})| \\ &= dim_{\mathbf{Q}} A_0(N) \times dim_{\mathbf{Q}} A_0(M) \times |L(N) \cap L(M)|. \end{aligned}$$

If $\lambda, \mu \in SC$, and $\lambda \neq \mu$, then $L(M) = L(\chi^{\lambda,\mu}) = \{1, \varepsilon\}$.

Consider the group pair $(W_n[+1, +1, +1, +1, +1], H)$, where H is the kernel of ε. By Corollary 3.5.8, $A := End_{\mathbf{Q}H}(N)$ with $A_0 = A_0(M)$. is a **CSGA** of odd type.

By Theorem 3.7.8, we have

$$\begin{aligned} A_0(W) &\simeq A_0(M) \otimes (S_M^2, \varepsilon_1) \otimes \mathbf{Q}(\sqrt{\varepsilon_2}) \\ &\simeq \mathbf{Q} \otimes (\varepsilon_1, Z_\lambda^* Z_\mu^*) \otimes \mathbf{Q}(\sqrt{\varepsilon_2}) \end{aligned}$$

since $S_M^2 = Z_\lambda^* Z_\mu^*$, where S_M is an ε-associator of M.

It then follows from Theorem 2.6.2 that

$$A^{\lambda,\mu}(\alpha) = (\varepsilon_1, Z_\lambda^* Z_\mu^*)$$

in $Br\left(\mathbf{Q}(\chi^{\lambda,\mu})\right)$ with $\mathbf{Q}(\chi^{\lambda,\mu}) = \mathbf{Q}(\sqrt{\varepsilon_2})$.

Then conclusion (1) is proved.

If $\lambda = \mu$, but not both of $\lambda, \mu \in SC$, then $L(M) = L(\chi^{\lambda,\mu}) = \{1, \delta\}$. By Theorem 3.7.8

$$\begin{aligned} A_0(W) &\simeq A_0(M) \otimes (T_M^2, \varepsilon_2) \otimes \mathbf{Q}(\sqrt{\varepsilon_1}) \\ &\simeq \mathbf{Q} \otimes (\varepsilon_2, 1) \otimes \mathbf{Q}(\sqrt{\varepsilon_1}) \end{aligned}$$

since $T_M^2 = 1$, where T_M is a δ-associator of M.

It then follows from Theorem 2.6.2 that

$$A^{\lambda,\mu}(\alpha) = 1$$

in $Br\left(\mathbf{Q}(\chi^{\lambda,\mu})\right)$ with $\mathbf{Q}(\chi^{\lambda,\mu}) = \mathbf{Q}(\sqrt{\varepsilon_1})$.

Then conclusion (2) is proved.

If $\lambda = \mu'$, but not both of $\lambda, \mu \in SC$, then $L(M) = L(\chi^{\lambda,\mu}) = \{1, \varepsilon\delta\}$. By Theorem 3.7.8

$$\begin{aligned} A_0(W) &\simeq A_0(M) \otimes (U_M^2, \varepsilon_2) \otimes \mathbf{Q}(\sqrt{\varepsilon_1\varepsilon_2}) \\ &\simeq \mathbf{Q} \otimes (\varepsilon_2, (-1)^{\frac{n}{2}}) \otimes \mathbf{Q}(\sqrt{\varepsilon_1\varepsilon_2}) \end{aligned}$$

since $U_M^2 = (-1)^{\frac{n}{2}}$, where U_M is an $\varepsilon\delta$-associator of M.

It then follows from Theorem 2.6.2 that

$$A^{\lambda,\mu}(\alpha) = \left(\varepsilon_2, (-1)^{n/2}\right)$$

in $Br\left(\mathbf{Q}(\chi^{\lambda,\mu})\right)$ with $\mathbf{Q}(\chi^{\lambda,\mu}) = \mathbf{Q}(\sqrt{\varepsilon_1\varepsilon_2})$.

Then conclusion (3) is proved.

Finally consider the case $\lambda = \mu \in SC$. We have $\mathbf{Q}(\chi^{\lambda,\mu}) = \mathbf{Q}$ and $L(M) = L(\chi^{\lambda,\mu}) = \{1, \varepsilon, \delta, \varepsilon\delta\}$. Let S, T be an ε-associator, δ-associator of M, respectively.

If $n \equiv 0$ $(mod\ 4)$, then by Theorem 3.1.7 $ST = TS$. By Theorem 3.7.8

$$\begin{aligned} A_0(W) &\simeq A_0(M) \otimes (S^2, \varepsilon_1) \otimes (T^2, \varepsilon_2) \\ &\simeq \mathbf{Q} \otimes (\varepsilon_1, Z_\lambda^* Z_\lambda^*) \otimes (\varepsilon_2, 1) \end{aligned}$$

since $T^2 = 1$ and $S^2 = Z_\lambda^* Z_\lambda^*$.

It then follows from Theorem 2.6.2 and Proposition 2.5.1 that

$$A^{\lambda,\mu}(\alpha) = 1$$

in $Br\left(\mathbf{Q}(\chi^{\lambda,\mu})\right)$ with $\mathbf{Q}(\chi^{\lambda,\mu}) = \mathbf{Q}$.

Then conclusion (4) is proved.

If $n \equiv 2$ $(mod\ 4)$, then by Theorem 3.1.7 $ST = -TS$. By Theorem 3.7.8

$$\begin{aligned} A_0(W) &\simeq A_0(M) \otimes (S^2\varepsilon_2, \varepsilon_1) \otimes (T^2, \varepsilon_2) \\ &\simeq \mathbf{Q} \otimes (\varepsilon_2 Z_\lambda^* Z_\lambda^*, \varepsilon_1) \otimes (1, \varepsilon_2). \end{aligned}$$

It then follows from Theorem 2.6.2 and Proposition 2.5.1 that

$$A^{\lambda,\mu}(\alpha) = (\varepsilon_1, \varepsilon_2)$$

in $Br\left(\mathbf{Q}(\chi^{\lambda,\mu})\right)$ with $\mathbf{Q}(\chi^{\lambda,\mu}) = \mathbf{Q}$.

Then conclusion (5) is proved.

For all other cases, i.e., $|L(M)| = |L(\chi^{\lambda,\mu})| = 1$, by Theorem 3.7.8

$$\begin{aligned} A_0(W) &\simeq A_0(N) \otimes A_0(M) \\ &\simeq A_0(N) \otimes \mathbf{Q} \\ &\simeq \begin{cases} \mathbf{Q} \oplus \mathbf{Q} \oplus \mathbf{Q} \oplus \mathbf{Q} & \text{if } \varepsilon_1 = 1 = \varepsilon_2 \\ \mathbf{Q}(\sqrt{\varepsilon_1}, \sqrt{\varepsilon_2}) \oplus \mathbf{Q}(\sqrt{\varepsilon_1}, \sqrt{\varepsilon_2}) & \text{otherwise .} \end{cases} \end{aligned}$$

It then follows from Theorem 2.6.2 that

$$A^{\lambda,\mu}(\alpha) = 1$$

in $Br\left(\mathbf{Q}(\chi^{\lambda,\mu})\right)$ with $\mathbf{Q}(\chi^{\lambda,\mu}) = \mathbf{Q}$ if $\varepsilon_1 = 1 = \varepsilon_2$, $\mathbf{Q}(\sqrt{-1})$ otherwise.

Then conclusion (6) is proved.

The proof is now complete. ■

4.3 $[\varepsilon_1, \varepsilon_2, -1, +1, +1]$

In this section, we study the spin characters of the double cover $W_n[\varepsilon_1, \varepsilon_2, -1, +1, +1]$. Theorem 3.5.2 shows that $\varphi^{\lambda,\mu}$ are all spin representations of the group $W_n[+1, +1, -1, +1, +1]$. As we have seen that $\varphi^{\lambda,\mu}$ does not take into account the first two components of the factor set $[\varepsilon_1, \varepsilon_2, -1, +1, +1]$, we will take care of the factor set $[\varepsilon_1, \varepsilon_2, -1, +1, +1]$ in the following study.

Let $\rho = \rho^{\varepsilon_1,\varepsilon_2}$ be the sum of the four linear spin representations of the group $W_n[\varepsilon_1, \varepsilon_1, +1, +1, +1]$. By Theorem 2.4.3, the constituents of the twisted product $\rho\varphi^{\lambda,\mu}$ form all the spin representations of the group $W_n[\varepsilon_1, \varepsilon_2, -1, +1, +1]$. We summarize this in the following theorem.

Theorem 4.3.1 *The irreducible spin representation of $W_n[\varepsilon_1, \varepsilon_2, -1, +1, +1]$ can be indexed so that the submodules of $\rho\Phi^{\lambda,\mu}$ are labeled by the unordered pair $\{\lambda, \mu\}$, where $\lambda, \mu \in DP$. In the following table, $n_{\lambda,\mu}$ denotes the number of modules indexed by (λ, μ), $m_{\lambda,\mu}$ denotes their multiplicity in $\rho\varphi^{\lambda,\mu}$, and $o_{\lambda,\mu}$ denotes the size of the orbit of $\{\lambda, \mu\}$.*

$n_{\lambda,\mu}$	$o_{\lambda,\mu}$	$m_{\lambda,\mu}$	
1	1	4	if $\lambda = \mu$
2	2	2	if $\lambda, \mu \in DP^+$ *or* $\lambda, \mu \in DP^-$ *and* $\lambda \neq \mu$
4	4	1	if $\lambda \in DP^+, \mu \in DP^-$ *or* $\lambda \in DP^-, \mu \in DP^+$.

Proof: The results follow from Theorem 3.4.1, Theorem 3.4.4 and Theorem 2.4.3. ∎

Theorem 4.3.2 *Let $\alpha = [\varepsilon_1, \varepsilon_2, -1, +1, +1]$, let $\chi^{\lambda,\mu}$ be any character indexed by λ, μ in the above theorem, and let $A^{\lambda,\mu}(\alpha)$, as an element of $Br\left(\mathbf{Q}(\chi^{\lambda,\mu})\right)$, be the central simple algebra associated with character $\chi^{\lambda,\mu}$. Recall that Definition 3.5.17 gives us $\mathcal{D}_0^{\lambda,\mu}, \mathcal{D}_1^{\lambda,\mu}, \mathcal{D}_2^{\lambda,\mu}, d_0^{\lambda,\mu}, d_1^{\lambda,\mu}$ and $d_2^{\lambda,\mu}$. Then we have the following:*

(1) If $\lambda \in DP^-, \mu \in DP^+$ or $\lambda \in DP^+, \mu \in DP^-$, then

$$A^{\lambda,\mu}(\alpha) = \mathcal{D}_0^{\lambda,\mu},$$

with $\mathbf{Q}(\chi^{\lambda,\mu}) = \mathbf{Q}\left(\sqrt{\varepsilon_1 d_0^{\lambda,\mu}}, \sqrt{\varepsilon_2}\right)$.

(2) If $\lambda, \mu \in DP^+$ *or* $\lambda, \mu \in DP^-$ *and* $\lambda \neq \mu$, *then*

$$A^{\lambda,\mu}(\alpha) \simeq \mathcal{D}_1^{\lambda,\mu}(d_1^{\lambda,\mu}, \varepsilon_1),$$

with $\mathbf{Q}(\chi^{\lambda,\mu}) = \mathbf{Q}(\sqrt{\varepsilon_2})$.

(3) If $\lambda = \mu$ *and* $l(\lambda)$ *is odd, then*

$$A^{\lambda,\mu}(\alpha) = \mathcal{D}_1^{\lambda,\mu}(\varepsilon_1, \varepsilon_2)(\varepsilon_1, d_1^{\lambda,\mu})(\varepsilon_2, d_2^{\lambda,\mu}),$$

with $\mathbf{Q}(\chi^{\lambda,\mu}) = \mathbf{Q}$.

(4) If $\lambda = \mu$ *and* $l(\lambda)$ *is even, then*

$$A^{\lambda,\mu}(\alpha) = \mathcal{D}_1^{\lambda,\mu}(\varepsilon_1, d_1^{\lambda,\mu})(\varepsilon_2, d_2^{\lambda,\mu}),$$

with $\mathbf{Q}(\chi^{\lambda,\mu}) = \mathbf{Q}$.

Proof: Let $G = W_n[\varepsilon_1, \varepsilon_2, -1, +1, +1]$, and let N be a $\mathbf{Q}W_n[\varepsilon_1, \varepsilon_2, +1, +1, +1]$-module affording ρ. We divide our discussion into several cases.

Case 1. $\varepsilon(\lambda, \mu) = -1$, i.e., $\lambda \in DP^-, \mu \in DP^+$ or $\lambda \in DP^+, \mu \in DP^-$.

Note from Corollary 3.5.16, $\varphi^{\lambda,\mu}$ has Schur index at most 2. Let M be a $\mathbf{Q}W_n[+1, +1, -1, +1, +1]$-module affording $2(\varphi_+^{\lambda,\mu} + \varphi_-^{\lambda,\mu})$, then $W = M \otimes N$, as a $\mathbf{Q}G$-module, affords

$$2(\varphi_+^{\lambda,\mu} + \varphi_-^{\lambda,\mu})\varrho = 4(\varphi_+^{\lambda,\mu})\varrho.$$

We want to calculate $A_0(W) := End_{\mathbf{Q}G}(W)$. By Corollary 3.4.3 we have

$$\begin{aligned} dim_{\mathbf{Q}} A_0(W) &= \|4(\varphi_+^{\lambda,\mu})\varrho\|^2 \\ &= 4^2 \|\varphi_+^{\lambda,\mu}\varrho\|^2 \\ &= 4^2 \times 4|L(\varphi_+^{\lambda,\mu})| \\ &= 4^3. \end{aligned}$$

On the other hand, we have

$$dim_{\mathbf{Q}}A_0(M) = \|2(\varphi_+^{\lambda,\mu} + \varphi_-^{\lambda,\mu})\|^2 = 2^2\|\varphi_+^{\lambda,\mu} + \varphi_-^{\lambda,\mu}\|^2 = 4 \times 2.$$

Notice that $dim_{\mathbf{Q}}A_0(N) = 4$, $L(N) = \{1, \varepsilon, \delta, \varepsilon\delta\}$, and $L(M) = \{1, \varepsilon\}$. We have

$$dim_{\mathbf{Q}}A_0(W) = dim_{\mathbf{Q}}A_0(N) \times dim_{\mathbf{Q}}A_0(M) \times |L(N) \cap L(M)|.$$

Consider the group pair $(W_n[+1, +1, -1, +1, +1], H)$, where H is the kernel of ε. By Corollary 3.5.8, $A := End_{\mathbf{Q}H}(M)$ is a **CSGA** of even type. Let

$$\begin{aligned} A &\simeq [\mathcal{D}_0^{\lambda,\mu}, 0, d_0^{\lambda,\mu}], A_0 = A_0(M) \\ Z(A_0) &\simeq \mathbf{Q} + i\mathbf{Q}, i^2 = d_0^{\lambda,\mu} \in \mathbf{Q}^{\times}. \end{aligned}$$

By Corollary 3.5.15, $\mathcal{D}_0^{\lambda,\mu}$ can be chosen to be a quaternion algebra. It is easy to check $dim_{\mathbf{Q}}A = 2 \times dim_{\mathbf{Q}}A_0$. Therefore Theorem 3.7.8 can be applied. Now $L(M) = \{1, \varepsilon\}$ and we are in the second situation of Theorem 3.7.8. We have

$$A_0(W) \simeq A \otimes R$$

where R is is a 4-dimensional abelian algebra which is a direct sum of fields and each of its summands is isomorphic to $\mathbf{Q}\left(\sqrt{\varepsilon_1 d_0^{\lambda,\mu}}, \sqrt{\varepsilon_2}\right)$.

It then follows from Theorem 2.6.2

$$A^{\lambda,\mu}(\alpha) = \mathcal{D}_0^{\lambda,\mu}$$

in $Br\left(\mathbf{Q}(\chi^{\lambda,\mu})\right)$ with $\mathbf{Q}(\chi^{\lambda,\mu}) = \mathbf{Q}\left(\sqrt{\varepsilon_1 d_0^{\lambda,\mu}}, \sqrt{\varepsilon_2}\right)$.

Case 2. $\varepsilon(\lambda, \mu) = 1$. i.e., $\lambda,\ \mu \in DP^+$ or $\lambda,\ \mu \in DP^-$.

Note from Corollary 3.5.16, $\varphi^{\lambda,\mu}$ has Schur index at most 2. Let M be a $\mathbf{Q}W_n[+1, +1, -1, +1, +1]$-module affording $2\varphi^{\lambda,\mu}$, then $W = M \otimes N$, as a $\mathbf{Q}G$-module, affords $2(\varphi^{\lambda,\mu})\varrho$. We want to calculate $A_0(W) := End_{\mathbf{Q}G}(W)$. By Corollary 3.4.3 we have

$$dim_{\mathbf{Q}}A_0(W) \quad = \quad \|2\varphi^{\lambda,\mu}\varrho\|^2$$

$$
\begin{aligned}
&= 2^2\|\varphi^{\lambda,\mu}\varrho\|^2 \\
&= 4 \times 4|L(\varphi^{\lambda,\mu})| \\
&= 4^2|L(\varphi^{\lambda,\mu})| \\
&= 4^2 \times 2.
\end{aligned}
$$

Notice $dim_{\mathbf{Q}}A_0(N) = 4$, $dim_{\mathbf{Q}}A_0(M) = 4$, $L(N) = \{1, \varepsilon, \delta, \varepsilon\delta\}$ and $L(M) = \{1, \varepsilon\}$. We have

$$dim_{\mathbf{Q}}A_0(W) = dim_{\mathbf{Q}}A_0(N) \times dim_{\mathbf{Q}}A_0(M) \times |L(N) \cap L(M)|.$$

First consider the case $\lambda \neq \mu$. We have $L(M) = L(\varphi^{\lambda,\mu}) = \{1, \varepsilon\}$.

Consider the group pair $(W_n[+1, +1, -1, +1, +1], H)$, where H is the kernel of ε. By Corollary 3.5.8, $A := End_{\mathbf{Q}H}(M)$ is a **CSGA** of odd type. Let

$$
\begin{aligned}
A &\simeq [\mathcal{D}_1^{\lambda,\mu}, 1, d_1^{\lambda,\mu}], A_0 = A_0(M) \\
Z(A) &\simeq \mathbf{Q} + i\mathbf{Q},\ i^2 = d_1^{\lambda,\mu} \in \mathbf{Q}^\times.
\end{aligned}
$$

By Theorem 3.7.8, we have

$$
\begin{aligned}
A_0(W) &\simeq A_0(M) \otimes (d_1^{\lambda,\mu}, \varepsilon_1) \otimes \mathbf{Q}(\sqrt{\varepsilon_2}) \\
&\simeq \mathcal{D}_1^{\lambda,\mu} \otimes (\varepsilon_1, d_1^{\lambda,\mu}) \otimes \mathbf{Q}(\sqrt{\varepsilon_2}).
\end{aligned}
$$

It then follows from Theorem 2.6.2 that

$$A^{\lambda,\mu}(\alpha) = \mathcal{D}_1^{\lambda,\mu}(\varepsilon_1, d_1^{\lambda,\mu})$$

in $Br\left(\mathbf{Q}(\chi^{\lambda,\mu})\right)$ with $\mathbf{Q}(\chi^{\lambda,\mu}) = \mathbf{Q}(\sqrt{\varepsilon_2})$.

Next let us attack the case $\lambda = \mu$, i.e., $L(\varphi^{\lambda,\mu}) = \{1, \varepsilon, \delta, \varepsilon\delta\}$.

Consider group pairs $(W_n[+1, +1, -1, +1, +1], H_i), i = 1, 2$, where H_1 is the kernel of ε, and H_2 is the kernel of δ. By Corollary 3.5.8, both graded algebras $A = A_{H_1}(M)$ and $B = A_{H_2}(M)$ are **CSGA** of odd type. Let

$$
\begin{aligned}
A &\simeq [\mathcal{D}_1^{\lambda,\mu}, 1, d_1^{\lambda,\mu}], A_0 = A_0(M) \\
Z(A) &\simeq \mathbf{Q} + i\mathbf{Q},\ i^2 = d_1^{\lambda,\mu} \in \mathbf{Q}^\times,
\end{aligned}
$$

and

$$\begin{aligned} B &= [\mathcal{D}_2^{\lambda,\mu}, 1, d_2^{\lambda,\mu}], B_0 = A_0(M) \\ Z(B) &= \mathbf{Q} + j\mathbf{Q},\ j^2 = d_2^{\lambda,\mu} \in \mathbf{Q}^\times. \end{aligned}$$

Actually, i is an ε-associator of M and j is a δ-associator of M.

If $l(\lambda)$ is odd, by Theorem 3.5.9, $ij = -ji$. By Theorem 3.7.8, we have

$$\begin{aligned} A_0(W) &\simeq A_0 \otimes (d_1^{\lambda,\mu}\varepsilon_2, \varepsilon_1) \otimes (\varepsilon_2, d_2^{\lambda,\mu}) \\ &\simeq \mathcal{D}_1^{\lambda,\mu} \otimes (d_1^{\lambda,\mu}\varepsilon_2, \varepsilon_1) \otimes (\varepsilon_2, d_2^{\lambda,\mu}). \end{aligned}$$

It then follows from Theorem 2.6.2 and Proposition 2.5.1 that

$$A^{\lambda,\mu}(\alpha) = \mathcal{D}_1^{\lambda,\mu}(\varepsilon_1, \varepsilon_2)(\varepsilon_1, d_1^{\lambda,\mu})(\varepsilon_2, d_2^{\lambda,\mu})$$

in $Br\left(\mathbf{Q}(\chi^{\lambda,\mu})\right)$ with $\mathbf{Q}(\chi^{\lambda,\mu}) = \mathbf{Q}$.

If $l(\lambda)$ is even, by Theorem 3.5.9, $ij = ji$. By Theorem 3.7.8, we have

$$\begin{aligned} A_0(W) &\simeq A_0 \otimes (d_1^{\lambda,\mu}, \varepsilon_1) \otimes (\varepsilon_2, d_2^{\lambda,\mu}) \\ &\simeq \mathcal{D}_1^{\lambda,\mu} \otimes (\varepsilon_1, d_1^{\lambda,\mu}) \otimes (\varepsilon_2, d_2^{\lambda,\mu}). \end{aligned}$$

It then follows from Theorem 2.6.2 taht

$$A^{\lambda,\mu}(\alpha) = \mathcal{D}_1^{\lambda,\mu}(\varepsilon_1, d_1^{\lambda,\mu})(\varepsilon_2, d_2^{\lambda,\mu})$$

in $Br\left(\mathbf{Q}(\chi^{\lambda,\mu})\right)$ with $\mathbf{Q}(\chi^{\lambda,\mu}) = \mathbf{Q}$.

Now the proof of the theorem is complete. ∎

4.4 $[\varepsilon_1, \varepsilon_2, -1, -1, +1]$

In this Section, we will study the irreducible spin representations of the double covers $W_n[\varepsilon_1, \varepsilon_2, -1, -1, +1]$. With the knowledge of $\Phi^{\lambda,\mu}$, according to Theorem 2.4.3, we are able to characterize the spin representations of $W_n[\varepsilon_1, \varepsilon_2, -1, -1, +1]$ in terms of $\Phi^{\lambda,\mu}$ and $\Theta^{\varepsilon_1,\varepsilon_2}$.

Theorem 4.4.1 *The irreducible spin representations of* $W_n[\varepsilon_1, \varepsilon_2, -1, -1, +1]$ *can be indexed so that the submodule of* $\Theta^{\varepsilon_1,\varepsilon_2}\Phi^{\lambda,\mu}$ *are labeled by the unordered pair* $\{\lambda, \mu\}$, *where* $\lambda, \mu \in DP$, *and* $|\lambda| + |\mu| = n$. *In the following table,* $n_{\lambda,\mu}$ *denotes the number of modules indexed by* (λ, μ), $m_{\lambda,\mu}$ *denotes their multiplicity in* $\Theta^{\varepsilon_1,\varepsilon_2}\Phi^{\lambda,\mu}$, *and* $o_{\lambda,\mu}$ *denotes the size of the orbit of* (λ, μ).

$n_{\lambda,\mu}$	$o_{\lambda,\mu}$	$m_{\lambda,\mu}$	
1	1	4	if $\lambda = \mu$, and $l(\lambda)$ is even
4	1	2	if $\lambda = \mu$, and $l(\lambda)$ is odd
2	2	2	if $n - l(\lambda) - l(\mu)$ is even and $\lambda \neq \mu$
1	4	2	if $n - l(\lambda) - l(\mu)$ is odd.

Proof: By Proposition 3.3.3 we have $\Theta^{\varepsilon_1,\varepsilon_2} = 2\Theta_0^{\varepsilon_1,\varepsilon_2}$. Then the case $\varepsilon_1 = 1 = \varepsilon_2$ follows readily from Corollary 8.5 of Stembridge [23]. The remaining cases follow from Stembridge's argument, using Theorem 3.3.5 and Theorem 2.4.3. ∎

Theorem 4.4.2 *Let* $\alpha = [\varepsilon_1, \varepsilon_2, -1, -1, +1]$, $\chi^{\lambda,\mu}$ *be any character indexed by the unordered pair* (λ, μ) *in the above Theorem, and let* $A^{\lambda,\mu}(\alpha)$, *as an element of* $Br(\mathbf{Q}(\chi^{\lambda,\mu}))$, *be the central simple algebra associated with the character* $\chi^{\lambda,\mu}$. *Recall that Definition 3.5.17 gives us* $\mathcal{D}_0^{\lambda,\mu}, \mathcal{D}_1^{\lambda,\mu}, \mathcal{D}_2^{\lambda,\mu}, d_0^{\lambda,\mu}, d_1^{\lambda,\mu}$ *and* $d_2^{\lambda,\mu}$. *Then*

(1) If $\lambda \in DP^-, \mu \in DP^+$ *or* $\lambda \in DP^+, \mu \in DP^-$, *then*

$$A^{\lambda,\mu}(\alpha) = (\varepsilon_2, \varepsilon_1 d_0^{\lambda,\mu})\mathcal{D}_0^{\lambda,\mu},$$

with $\mathbf{Q}(\chi) = \mathbf{Q}$.

(2) If $\lambda, \mu \in DP^+$ *or* $\lambda, \mu \in DP^-$ *and if* $\lambda \neq \mu$, *then*

$$A^{\lambda,\mu}(\alpha) = (\varepsilon_1, \varepsilon_2)\mathcal{D}_1^{\lambda,\mu},$$

with $\mathbf{Q}(\chi) = \mathbf{Q}\left(\sqrt{\varepsilon_2 d_1^{\lambda,\mu}}\right)$.

(3) If $\lambda = \mu$ *and* $l(\lambda)$ *is odd, then*

$$A^{\lambda,\mu}(\alpha) = (\varepsilon_1, \varepsilon_2)\mathcal{D}_1^{\lambda,\mu},$$

with $\mathbf{Q}(\chi) = \mathbf{Q}\left(\sqrt{\varepsilon_2 d_1^{\lambda,\mu}}, \sqrt{\varepsilon_1 d_2^{\lambda,\mu}}\right)$.

(4) If $\lambda = \mu$ and $l(\lambda)$ is even, then

$$A^{\lambda,\mu}(\alpha) = (\varepsilon_2 d_1^{\lambda,\mu}, \varepsilon_1 d_2^{\lambda,\mu})\mathcal{D}_1^{\lambda,\mu},$$

with $\mathbf{Q}(\chi) = \mathbf{Q}$.

Proof: Let $G = W_n[\varepsilon_1, \varepsilon_2, -1, -1, +1]$. By Corollary 3.5.16, the Schur indices of $\varphi^{\lambda,\mu}$ are always one or two and there is a $\mathbf{Q}W_n[+1, +1, -1, +1, +1]$-module M affording $2\varphi^{\lambda,\mu}$ (when $\varepsilon(\lambda, \mu) = 1$) or $2(\varphi_+^{\lambda,\mu} + \varphi_-^{\lambda,\mu})$ (when $\varepsilon(\lambda, \mu) = -1$). Let N be a $\mathbf{Q}W_n[\varepsilon_1, \varepsilon_2, +1, -1, +1]$-module affording $\theta^{\varepsilon_1,\varepsilon_2}$, then $W = N \otimes M$, as a $\mathbf{Q}W_n[\varepsilon_1, \varepsilon_2, -1, -1, +1]$-module, affords the twisted product

$$\begin{aligned}
\theta^{\varepsilon_1,\varepsilon_2}\left(2\varphi^{\lambda,\mu}\right) &= 2(\theta^{\varepsilon_1,\varepsilon_2}\varphi^{\lambda,\mu}) \text{ if } \varepsilon(\lambda,\mu) = 1. \\
\theta^{\varepsilon_1,\varepsilon_2} 2\left(\varphi_+^{\lambda,\mu} + \varphi_-^{\lambda,\mu}\right) &= 4(\theta^{\varepsilon_1,\varepsilon_2}\varphi_+^{\lambda,\mu}) \text{ if } \varepsilon(\lambda,\mu) = -1.
\end{aligned}$$

Our purpose is to investigate the structure of $A_0(W) := End_{\mathbf{Q}G}(W)$.

Case 1. $\varepsilon(\lambda, \mu) = -1$, i.e., $\lambda \in DP^-, \mu \in DP^+$ or $\lambda \in DP^+, \mu \in DP^-$.

In this case, $L(\varphi_+^{\lambda,\mu}) = \{1\}$ and $L(M) = \{1, \varepsilon\}$.

By Proposition 3.3.4, we have

$$\begin{aligned}
dim_{\mathbf{Q}} A_0(W) &= \|4(\theta^{\varepsilon_1,\varepsilon_2}\varphi_+^{\lambda,\mu})\|^2 \\
&= 4^2\|\theta^{\varepsilon_1,\varepsilon_2}\varphi_+^{\lambda,\mu}\|^2 \\
&= 4^2 \times 4 \times |L(\varphi_+^{\lambda,\mu})| \\
&= 4^3.
\end{aligned}$$

Notice that $A_0(N) \simeq (\varepsilon_1, \varepsilon_2)$, $dim_{\mathbf{Q}} A_0(M) = 4 \times 2$, $L(N) = \{1, \varepsilon, \delta, \varepsilon\delta\}$, and $L(M) = \{1, \varepsilon\}$. We have

$$dim_{\mathbf{Q}} A_0(W) = dim_{\mathbf{Q}} A_0(N) \times dim_{\mathbf{Q}} A_0(M) \times |L(N) \cap L(M)|.$$

Consider the group pair $(W_n[+1, +1, -1, +1, +1], H)$, where H is the kernel of ε. By Corollary 3.5.8, $A := End_{\mathbf{Q}H}(M)$ is a **CSGA** of even type. Let

$$A \simeq [\mathcal{D}_0^{\lambda,\mu}, 0, d_0^{\lambda,\mu}], A_0 = A_0(M)$$

$$Z(A_0) \simeq \mathbf{Q} + i\mathbf{Q}, i^2 = d_0^{\lambda,\mu} \in \mathbf{Q}^\times.$$

By Corollary 3.5.15, $\mathcal{D}_0^{\lambda,\mu}$ can be chosen to be a quaternion algebra. It is easy to check that $dim_{\mathbf{Q}} A = 2 \times dim_{\mathbf{Q}} A_0$. By Theorem 3.7.7, we have

$$A_0(W) \simeq (\varepsilon_1, \varepsilon_2) \otimes A_H(M) \otimes (d_0^{\lambda,\mu}, \varepsilon_2).$$

It then follows from Theorem 2.6.2 and Proposition 2.5.1 that

$$A^{\lambda,\mu}(\alpha) = \mathcal{D}_0^{\lambda,\mu}(\varepsilon_1 d_0^{\lambda,\mu}, \varepsilon_2)$$

in $Br\left(\mathbf{Q}(\chi^{\lambda,\mu})\right)$ with $\mathbf{Q}(\chi^{\lambda,\mu}) = \mathbf{Q}$.

Case 2. $\varepsilon(\lambda,\mu) = 1$ i.e., $\lambda, \mu \in DP^-$, or $\lambda, \mu \in DP^+$

In this case M affords $2\varphi^{\lambda,\mu}$ and $\mathbf{Q}(\varphi^{\lambda,\mu}) = \mathbf{Q}$. And

$$\begin{aligned} dim_{\mathbf{Q}} A_0(W) &= ||2(\theta^{\varepsilon_1,\varepsilon_2}\varphi^{\lambda,\mu})||^2 \\ &= 2^2||\theta^{\varepsilon_1,\varepsilon_2}\varphi^{\lambda,\mu}||^2 \\ &= 4^2|L(\varphi^{\lambda,\mu})| \\ &= 4^2 \times 2. \end{aligned}$$

Notice that $A_0(N) \simeq (\varepsilon_1, \varepsilon_2)$, $dim_{\mathbf{Q}} A_0(M) = 4$, $L(N) = \{1, \varepsilon, \delta, \varepsilon\delta\}$, and $L(M) = \{1, \varepsilon\}$. We have

$$dim_{\mathbf{Q}} A_0(W) = dim_{\mathbf{Q}} A_0(N) \times dim_{\mathbf{Q}} A_0(M) \times |L(N) \cap L(M)|.$$

Suppose first $\lambda, \mu \in DP^-$, or $\lambda, \mu \in DP^+$ but $\lambda \neq \mu$. In this case, $L(M) = L(\varphi^{\lambda,\mu}) = \{1, \varepsilon\}$.

Consider the group pair $(W_n[+1, +1, -1, +1, +1], H)$, where H is the kernel of ε. By Corollary 3.5.8, $A := End_{\mathbf{Q}H}(M)$ is a **CSGA** of odd type. Let

$$\begin{aligned} A &\simeq [\mathcal{D}_1^{\lambda,\mu}, 1, d_1^{\lambda,\mu}], A_0 = A_0(M) \\ Z(A) &\simeq \mathbf{Q} + i\mathbf{Q},\ i^2 = d_1^{\lambda,\mu} \in \mathbf{Q}^\times. \end{aligned}$$

By Theorem 3.7.7, we have

$$\begin{aligned} A_0(W) &\simeq (\varepsilon_1, \varepsilon_2) \otimes A_0(M) \otimes \mathbf{Q}\left(\sqrt{\varepsilon_2 d_1^{\lambda,\mu}}\right) \\ &\simeq (\varepsilon_1, \varepsilon_2) \otimes \mathbf{Q} \otimes \mathbf{Q}\left(\sqrt{\varepsilon_2 d_1^{\lambda,\mu}}\right). \end{aligned}$$

It then follows from Theorem 2.6.2

$$A^{\lambda,\mu}(\alpha) = (\varepsilon_1, \varepsilon_2)\mathcal{D}_1^{\lambda,\mu}$$

in $Br\left(\mathbf{Q}(\chi^{\lambda,\mu})\right)$ with $\mathbf{Q}(\chi^{\lambda,\mu}) = \mathbf{Q}\left(\sqrt{\varepsilon_2 d_1^{\lambda,\mu}}\right)$.

Next suppose $\lambda = \mu$. In this case, we have $L(\varphi^{\lambda,\mu}) = L(M) = \{1, \varepsilon, \delta, \varepsilon\delta\}$.

Consider the group pairs $(W_n[\varepsilon_1, \varepsilon_2, -1, +1, +1], H_i), i = 1, 2$, where H_1 is the kernel of ε, and H_2 is the kernel of δ. By Corollary 3.5.8, both graded algebras $A = A_{H_1}(M)$ and $B = A_{H_2}(M)$ are **CSGA** of odd type. Let

$$\begin{aligned} A &\simeq [\mathcal{D}_1^{\lambda,\mu}, 1, d_1^{\lambda,\mu}],\ A_0 = A_0(M) \\ Z(A) &\simeq \mathbf{Q} + i\mathbf{Q},\ i^2 = d_1^{\lambda,\mu} \in \mathbf{Q}^\times \end{aligned}$$

where i can be chosen to be an ε-associator.

And

$$\begin{aligned} B &\simeq [\mathcal{D}_2^{\lambda,\mu}, 1, d_2^{\lambda,\mu}],\ B_0 = A_0(M) \\ Z(B) &\simeq \mathbf{Q} + j\mathbf{Q},\ j^2 = d_2^{\lambda,\mu} \in \mathbf{Q}^\times \end{aligned}$$

where j can be chosen to be a δ-associator.

Based on the possible cases $ij = ji$ or $ij = -ji$, we need to separate these two situations.

First assume $l(\lambda)$ is odd, by Theorem 3.5.9 we have $ij = -ji$. Then by Theorem 3.7.7, we get

$$\begin{aligned} A_0(W) &\simeq (\varepsilon_1, \varepsilon_2) \otimes A_0(M) \otimes R \\ &\simeq (\varepsilon_1, \varepsilon_2) \otimes \mathbf{Q} \otimes R \end{aligned}$$

where R is a direct sum of fields and each of summands is isomorphic to

$$\mathbf{Q}\left(\sqrt{\varepsilon_2 d_1^{\lambda,\mu}}, \sqrt{\varepsilon_1 d_2^{\lambda,\mu}}\right).$$

It then follows from Theorem 2.6.2 that

$$A^{\lambda,\mu}(\alpha) = (\varepsilon_1, \varepsilon_2)\mathcal{D}_1^{\lambda,\mu}$$

in $Br\left(\mathbf{Q}(\chi^{\lambda,\mu})\right)$ with $\mathbf{Q}(\chi^{\lambda,\mu}) = \mathbf{Q}\left(\sqrt{\varepsilon_2 d_1^{\lambda,\mu}}, \sqrt{\varepsilon_1 d_2^{\lambda,\mu}}\right)$.

Next assume $l(\lambda)$ is even, by Theorem 3.5.9 we have $ij = ji$. Then by Theorem 3.7.7, we get

$$A_0(W) \simeq (\varepsilon_1, \varepsilon_2) \otimes A_0(M) \otimes (d_1^{\lambda,\mu}\varepsilon_2, d_2^{\lambda,\mu}\varepsilon_1).$$

It then follows from Theorem 2.6.2 and Proposition 2.5.1 that

$$A^{\lambda,\mu}(\alpha) = (\varepsilon_2 d_1^{\lambda,\mu}, \varepsilon_1 d_2^{\lambda,\mu})\mathcal{D}_1^{\lambda,\mu}$$

in $Br\left(\mathbf{Q}(\chi^{\lambda,\mu})\right)$ with $\mathbf{Q}(\chi^{\lambda,\mu}) = \mathbf{Q}$.

The proof of the theorem is now complete. ■

4.5 $[\varepsilon_1, \varepsilon_2, -1, -1, -1]$

In this section, we will study the spin representations of the double covers $W_n[\varepsilon_1, \varepsilon_2, -1, -1, -1]$. With the knowledge of $\Psi^{\varepsilon_1,\varepsilon_2}$, we are able to characterize all spin representations of the double covers $W_n[\varepsilon_1, \varepsilon_2, -1, -1, -1]$ in terms of $\Psi^{\varepsilon_1,\varepsilon_2}$ and $X^{\lambda,\mu}$.

Theorem 4.5.1 *The irreducible spin representations of $W_n[\varepsilon_1, \varepsilon_2, -1, -1, -1]$ are $\Psi^{\varepsilon_1,\varepsilon_2}X^{\lambda,\emptyset}$ (for n even) and $\Psi_{\pm}^{\varepsilon_1,\varepsilon_2}X^{\lambda,\emptyset}$ (for n odd), i.e., the irreducible spin representations of $W_n[\varepsilon_1, \varepsilon_2, -1, -1, -1]$ can be indexed by λ, where λ ranges over the partitions of n.*

Proof: The case $\varepsilon_1 = 1 = \varepsilon_2$ follows readily from Theorem 9.2 of Stembridge [23]. The remaining cases follow from the same proof of Theorem 9.2 of [23]. ■

Theorem 4.5.2 *Let $\alpha = [\varepsilon_1, \varepsilon_2, -1, -1, -1]$, let χ^λ be any character indexed in the above theorem, and let $A^\lambda(\alpha)$ be the central simple algebra associated with the character χ^λ. Recall the definition of Z_λ^* in Definition 3.1.4. Then we have*

(1) If $\lambda \notin SC$ and $2 \not| n$, then

$$A^\lambda(\alpha) = (-1,-1)^{\binom{n+1}{4}},$$

with $\mathbf{Q}(\chi^\lambda) = \mathbf{Q}\left(\sqrt{-2\varepsilon_1(-1)^{\binom{n+1}{2}}}, \sqrt{2\varepsilon_1\varepsilon_2}\right)$.

(2) If $\lambda \notin SC$ and $2|n$, then

$$A^\lambda(\alpha) = (-1,-1)^{\binom{n+1}{4}}\left(2\varepsilon_1, (-1)^{n/2}\right),$$

with $\mathbf{Q}(\chi^\lambda) = \mathbf{Q}\left(\sqrt{2\varepsilon_1\varepsilon_2}\right)$.

(3) If $\lambda \in SC$ and $2 \not| n$, then

$$A^\lambda(\alpha) = (-1,-1)^{\binom{n+1}{4}}(2\varepsilon_1\varepsilon_2, Z_\lambda^*),$$

with $\mathbf{Q}(\chi^\lambda) = \mathbf{Q}\left(\sqrt{-\varepsilon_2(-1)^{\binom{n+1}{2}}}\right)$.

(4) If $\lambda \in SC$ and $2|n$, then

$$A^\lambda(\alpha) = (-1,-1)^{\binom{n+1}{4}}\left(\varepsilon_2, (-1)^{n/2}\right)\left(2\varepsilon_1\varepsilon_2, Z_\lambda^*\right),$$

with $\mathbf{Q}(\chi^\lambda) = \mathbf{Q}$.

Proof: Let $G = W_n[\varepsilon_1, \varepsilon_2, -1, -1, -1]$. Let M be a $\mathbf{Q}G$-module affording $\Im$ defined in Section 3.6 and N be a $\mathbf{Q}W_n[+1,+1,+1,+1,+1]$-module affording $X^{\lambda,\emptyset}$. Then $W = N \otimes M$, as a $\mathbf{Q}W_n[\varepsilon_1, \varepsilon_2, -1, -1, -1]$-module, affords the twisted product $X^{\lambda,\emptyset}\Im$. We want to compute $A_0(W) := End_{\mathbf{Q}G}(W)$. Denote $\psi^{\varepsilon_1,\varepsilon_2}$ by ψ.

To apply Theorems in Section 3.7, we need to check the condition

$$dim_{\mathbf{Q}}A_0(W) = dim_{\mathbf{Q}}A_0(N) \times dim_{\mathbf{Q}}A_0(M) \times |L(N) \cap L(M)|.$$

If $2 \not| n$, by Lemma 3.1.1 we have

$$\begin{aligned}
\Im X^{\lambda,\emptyset} &= 2(\psi + \varepsilon\psi + \delta\psi + \varepsilon\delta\psi)X^{\lambda,\emptyset} \\
&= 2(\psi X^{\lambda,\emptyset} + \varepsilon\psi X^{\lambda,\emptyset} + \delta\psi X^{\lambda,\emptyset} + \varepsilon\delta\psi X^{\lambda,\emptyset}) \\
&= \begin{cases} 2(\psi X^{\lambda,\emptyset} + \varepsilon\psi X^{\lambda,\emptyset} + \delta\psi X^{\lambda,\emptyset} + \varepsilon\delta\psi X^{\lambda,\emptyset}) & \text{if } \lambda \notin SC \\ 4(\psi X^{\lambda,\emptyset} + \delta\psi X^{\lambda,\emptyset}) & \text{if } \lambda \in SC. \end{cases}
\end{aligned}$$

It follows

$$dim_{\mathbf{Q}} A_0(W) = \begin{cases} 16 & \text{if } \lambda \notin SC \\ 32 & \text{if } \lambda \in SC. \end{cases}$$

If $2|n$, by Lemma 3.1.1 we have

$$\begin{aligned}
\Im X^{\lambda,\emptyset} &= 4(\psi + \varepsilon\psi)X^{\lambda,\emptyset} \\
&= 4(\psi X^{\lambda,\emptyset} + \varepsilon\psi X^{\lambda,\emptyset}) \\
&= \begin{cases} 4(\psi X^{\lambda,\emptyset} + \varepsilon\psi X^{\lambda,\emptyset}) & \text{if } \lambda \notin SC \\ 8\psi X^{\lambda,\emptyset} & \text{if } \lambda \in SC. \end{cases}
\end{aligned}$$

It follows that

$$dim_{\mathbf{Q}} A_0(W) = \begin{cases} 32 & \text{if } \lambda \notin SC \\ 64 & \text{if } \lambda \in SC. \end{cases}$$

On the other hand, by Corollary 3.6.12

$$dim_{\mathbf{Q}} A_0(M) = \begin{cases} 16 & \text{if } 2 \not| n \\ 32 & \text{if } 2|n. \end{cases}$$

Notice $L(M) = \{1, \varepsilon, \delta, \varepsilon\delta\}$ and

$$L(N) = \begin{cases} \{1\} & \text{if } 2 \not| n \\ \{1, \varepsilon\} & \text{if } 2|n. \end{cases}$$

Therefore, in any case we do have

$$dim_{\mathbf{Q}} A_0(W) = dim_{\mathbf{Q}} A_0(N) \times dim_{\mathbf{Q}} A_0(M) \times |L(N) \cap L(M)|.$$

Assume first $\lambda \notin SC$. In this case, $L(N) = \{1\}$. By Theorem 3.7.4

$$\begin{aligned}
A_0(W) &\simeq A_0(N) \otimes A_0(M) \\
&\simeq \mathbf{Q} \otimes A_0(M) \\
&\simeq A_0(M).
\end{aligned}$$

If $2 \not| n$, then by Theorem 3.6.13 we have

$$A_0(W) \simeq (-1,-1)^{\binom{n+1}{4}} \otimes \mathbf{Q}\left(\sqrt{-2\varepsilon_1(-1)^{\binom{n+1}{2}}}\right) \otimes \mathbf{Q}\left(\sqrt{2\varepsilon_1\varepsilon_2}\right).$$

It then follows from Corollary 2.8.5 and Theorem 2.6.2 that

$$A^{\lambda,\mu}(\alpha) = (-1,-1)^{\binom{n+1}{4}}$$

in $Br\left(\mathbf{Q}(\chi^{\lambda,\mu})\right)$ with $\mathbf{Q}(\chi^{\lambda,\mu}) = \mathbf{Q}\left(\sqrt{-2\varepsilon_1(-1)^{\binom{n+1}{2}}}, \sqrt{2\varepsilon_1\varepsilon_2}\right)$.

If $2|n$, then by Theorem 3.6.13 we have

$$A_0(W) \simeq \mathbf{Q} \otimes (-1,-1)^{\binom{n+1}{4}} \otimes \left(2\varepsilon_1, (-1)^{n/2}\right) \otimes \mathbf{Q}\left(\sqrt{2\varepsilon_1\varepsilon_2}\right).$$

It then follows from Theorem 2.6.2 that

$$A^{\lambda,\mu}(\alpha) = (-1,-1)^{\binom{n+1}{4}} \left(2\varepsilon_1, (-1)^{n/2}\right)$$

in $Br\left(\mathbf{Q}(\chi^{\lambda,\mu})\right)$ with $\mathbf{Q}(\chi^{\lambda,\mu}) = \mathbf{Q}\left(\sqrt{2\varepsilon_1\varepsilon_2}\right)$.

Next assume $\lambda \in SC$. In this case, $L(N) \cap L(M) = \{1, \varepsilon\}$.

Consider the group pair $(W_n[+1,+1,-1,+1,+1], H)$, where H is the kernel of ε. By Corollary 3.5.8, $A := End_{\mathbf{Q}H}(N)$ is a **CSGA** of odd type. Let

$$\begin{aligned} A &\simeq [D, 1, d], A_0 = A_0(N) \\ Z(A) &\simeq \mathbf{Q} + S_N\mathbf{Q}, S_N^2 = d \in \mathbf{Q}^\times \end{aligned}$$

where S_N is an ε-associator.

With the notations $S_M = S_M^\varepsilon, J_M$ of the Definition 3.8.5, by Corollary 3.7.5 we have

$$\begin{aligned} A_0(N \otimes M) &\simeq A_0(N) \otimes (J_M^2, S_N^2 S_M^2) \otimes C_{A_0(M)}(S_M) \\ &\simeq \mathbf{Q} \otimes (2\varepsilon_1\varepsilon_2, d) \otimes C_{A_0(M)}(S_M) \end{aligned}$$

If $2 \not| n$, then by Lemma 3.8.8 we get

$$\begin{aligned} A_0(W) \simeq\; & (2\varepsilon_1\varepsilon_2, d) \otimes (-1,-1)^{\binom{n+1}{4}} \otimes (1,1)^2 \\ & \otimes \mathbf{Q}\left(\sqrt{-\varepsilon_2(-1)^{\binom{n+1}{2}}}\right). \end{aligned}$$

It then follows from Theorem 2.6.2 that

$$A^{\lambda,\mu}(\alpha) = (-1,-1)^{\binom{n+1}{4}} (2\varepsilon_1\varepsilon_2, Z_\lambda^*)$$

in $Br\left(\mathbf{Q}(\chi^{\lambda,\mu})\right)$ with $\mathbf{Q}(\chi^{\lambda,\mu}) = \mathbf{Q}\left(\sqrt{-\varepsilon_2(-1)^{\binom{n+1}{2}}}\right)$.

If $2|n$, then by Lemma 3.8.8 we get

$$A_0(W) \simeq (-1,-1)^{\binom{n+1}{4}} \otimes (2\varepsilon_1\varepsilon_2, Z_\lambda^*) \otimes (1,1)^2 \otimes \left(\varepsilon_2, (-1)^{n/2}\right).$$

It then follows from Theorem 2.6.2 that

$$A^{\lambda,\mu}(\alpha) = (-1,-1)^{\binom{n+1}{4}} (2\varepsilon_1\varepsilon_2, Z_\lambda^*) \left(\varepsilon_2, (-1)^{n/2}\right)$$

in $Br\left(\mathbf{Q}(\chi^{\lambda,\mu})\right)$ with $\mathbf{Q}(\chi^{\lambda,\mu}) = \mathbf{Q}$.

The proof of the theorem is now complete. ∎

4.6 $[\varepsilon_1, \varepsilon_2, -1, +1, -1]$

In this section we will characterize all irreducible spin representations of the double cover $W_n[\varepsilon_1, \varepsilon_2, -1, +1, -1]$. Let $\Theta_0 = \Theta_0^{1,1}$ be the spin representation of $W_n[+1, +1, +1, -1, +1]$ defined in Section 2.4. By Theorem 2.4.3, irreducible spin characters of the double cover $W_n[\varepsilon_1, \varepsilon_2, -1, +1, -1]$ can be characterized in terms of Θ_0, $\Psi^{\varepsilon_1,\varepsilon_2}$ and $X^{\lambda,\mu}$. The next theorem tells exactly how.

Theorem 4.6.1 *The irreducible spin representations of $W_n[\varepsilon_1, \varepsilon_2, -1, +1, -1]$ can be labeled by $\mathbf{Z}_2$-orbits of the form $\{\lambda, \lambda'\}$, where λ ranges over partitions of n. The index $\{\lambda, \lambda'\}$ labels submodules of $\Theta_0\Psi^{\varepsilon_1,\varepsilon_2}X^{\lambda,\emptyset}$. In the following table, n_λ denotes the number of modules indexed by $\{\lambda, \lambda'\}$, m_λ denotes their multiplicity in $\Theta_0\Psi^{\varepsilon_1,\varepsilon_2}X^{\lambda,\emptyset}$, and o_λ denotes the size of the orbit of (λ, μ).*

n_λ	o_λ	m_λ	
1	1	2	if $\lambda \in SC$, and n is even
2	2	1	if $\lambda \in SC$, and n is odd or $\lambda \notin SC$ and n is even
1	4	1	if $\lambda \notin SC$, and n is odd .

Proof: By Proposition 3.3.3 we have $\Theta^{\varepsilon_1,\varepsilon_2} = 2\Theta_0^{\varepsilon_1,\varepsilon_2}$. Then the case $\varepsilon_1 = 1 = \varepsilon_2$ follows readily from Corollary 9.5 of Stembridge [23]. The remaining cases follow from the Stembridge's argument, using Theorem 3.3.5 and Theorem 2.4.3. ■

Theorem 4.6.2 *Let $\alpha = [\varepsilon_1, \varepsilon_2, -1, +1, -1]$, let χ^λ be any character indexed by λ in the above theorem, and let $A^\lambda(\alpha)$, as an element of $Br(\mathbf{Q}(\chi^{\lambda,\mu}))$, be the central simple algebra associated with the character χ^λ. Recall the definition of Z^*_λ in Definition 3.1.4. Then we have*

(1) If $\lambda \notin SC$ and $2 \not| n$, then

$$A^\lambda(\alpha) = (-1,-1)^{\binom{n+1}{4}} \left(-\varepsilon_2(-1)^{\binom{n+1}{2}}, -2\varepsilon_1\varepsilon_2\right),$$

with $\mathbf{Q}(\chi^\lambda) = \mathbf{Q}$.

(2) If $\lambda \notin SC$ and $2|n$, then

$$A^\lambda(\alpha) = (-1,-1)^{\binom{n+1}{4}} (\varepsilon_2, 2\varepsilon_1),$$

with $\mathbf{Q}(\chi^\lambda) = \mathbf{Q}\left(\sqrt{-2\varepsilon_1\varepsilon_2(-1)^{n/2}}\right)$.

(3) If $\lambda \in SC$ and $2 \not| n$, then

$$A^\lambda(\alpha) = (-1,-1)^{\binom{n+1}{4}} (2\varepsilon_1\varepsilon_2, Z^*_\lambda),$$

with $\mathbf{Q}(\chi^\lambda) = \mathbf{Q}\left(\sqrt{-\varepsilon_2(-1)^{\binom{n+1}{2}}}\right)$.

(4) If $\lambda \in SC$ and $2|n$, then

$$A^\lambda(\alpha) = (-1,-1)^{\binom{n+1}{4}} \left(\varepsilon_2, (-1)^{n/2}\right) (2\varepsilon_1\varepsilon_2, Z^*_\lambda),$$

with $\mathbf{Q}(\chi^\lambda) = \mathbf{Q}$.

Proof: Let $G = W_n[\varepsilon_1, \varepsilon_2, -1, +1, -1]$. Let N be a $\mathbf{Q}W_n[+1,+1,+1,+1,+1]$-module affording $X^{\lambda,\emptyset}$, let O be a $\mathbf{Q}W_n[+1,+1,+1,-1,+1]$-module affording θ_0, P be a $\mathbf{Q}W_n[\varepsilon_1, \varepsilon_2, -1, -1, -1]$-module affording $\Im$ defined in Section 3.6. Then $W = N \otimes M$, as a $\mathbf{Q}W_n[\varepsilon_1, \varepsilon_2, -1, +1, -1]$-module, affords the twisted product $X^{\lambda,\emptyset}\Theta_0\Im$,

where $M = O \otimes P$ affords the twisted product $\Theta_0\Im$. Our purpose is to calculate $A_0(W) := End_{\mathbf{Q}G}(W)$. Denote $\psi^{\varepsilon_1,\varepsilon_2}$ by ψ.

By Lemma 3.8.3 we have

$$dim_{\mathbf{Q}}A_0(W) = dim_{\mathbf{Q}}A_0(N) \times dim_{\mathbf{Q}}A_0(M) \times |L(N) \cap L(M)|,$$

and

$$dim_{\mathbf{Q}}A_0(M) = dim_{\mathbf{Q}}A_0(O) \times dim_{\mathbf{Q}}A_0(P) \times |L(O) \cap L(P)|.$$

Therefore when we apply theorems of Section 3.7, we do not need to worry about checking the dimensions.

Assume first $\lambda \notin SC$. In this case we have $L(N) = 1$. By Theorem 3.7.4, we have

$$\begin{aligned} A_0(N \otimes M) &\simeq A_0(N) \otimes A_0(M) \\ &\simeq \mathbf{Q} \otimes A_0(M) \\ &\simeq A_0(M). \end{aligned}$$

If $2 \not| n$, then by Theorem 3.8.6, we have

$$A_0(W) \simeq (-1,-1)^{\binom{n+1}{4}} \otimes (1,1)^2 \otimes \left(-\varepsilon_2(-1)^{\binom{n+1}{2}}, -2\varepsilon_1\varepsilon_2\right).$$

It then follows from Theorem 2.6.2 and Proposition 2.5.1 that

$$A^{\lambda,\mu}(\alpha) \simeq (-1,-1)^{\binom{n+1}{4}} \left(-\varepsilon_2(-1)^{\binom{n+1}{2}}, -2\varepsilon_1\varepsilon_2\right)$$

in $Br\left(\mathbf{Q}(\chi^{\lambda,\mu})\right)$ with $\mathbf{Q}(\chi^{\lambda,\mu}) = \mathbf{Q}$.

If $2|n$, then by Theorem 3.8.6, we have

$$\begin{aligned} A_0(W) \simeq\ & (-1,-1)^{\binom{n+1}{4}} \otimes (1,1)^2 \otimes (\varepsilon_2, 2\varepsilon_1) \\ & \otimes \mathbf{Q}\left(\sqrt{-2\varepsilon_1\varepsilon_2(-1)^{n/2}}\right). \end{aligned}$$

It then follows from Theorem 2.6.2 and that Proposition 2.5.1

$$A^{\lambda,\mu}(\alpha) \simeq (-1,-1)^{\binom{n+1}{4}}(\varepsilon_2, 2\varepsilon_1)$$

in $Br\left(\mathbf{Q}(\chi^{\lambda,\mu})\right)$ with $\mathbf{Q}(\chi^{\lambda,\mu}) = \mathbf{Q}\left(\sqrt{-2\varepsilon_1\varepsilon_2(-1)^{n/2}}\right)$.

Next assume $\lambda \in SC$. In this case $L(N) = \{1, \varepsilon\}$.

Consider the group pair $(W_n[+1,+1,+1,+1,+1], H)$, where H is the kernel of ε. By Corollary 3.5.8, $A := End_{\mathbf{Q}H}(M)$ is a **CSGA** of odd type. Let

$$\begin{aligned} A &\simeq [D,1,d], A_0 = A_0(N) \\ Z(A) &\simeq \mathbf{Q} + S_N\mathbf{Q}, S_N^2 = d \in \mathbf{Q}^{\times}. \end{aligned}$$

With the notations $S_N, S_M = S_M^{\varepsilon}$ of Definition 3.8.5, by Corollary 3.7.5 we have

$$\begin{aligned} A_0(N \otimes M) &\simeq A_0(N) \otimes (J_M^2, S_N^2 S_M^2) \otimes C_{A_0(M)}(S_M) \\ &\simeq \mathbf{Q} \otimes (J_M^2, dS_M^2) \otimes C_{A_0(M)}(S_M) \\ &\simeq (2\varepsilon_1\varepsilon_2, d) \otimes C_{A_0(M)}(S_M). \end{aligned}$$

To be specific, we distinguish the two cases between $2|n$ and $2 \not| n$.

If $2 \not| n$, by Lemma 3.8.7 we get

$$\begin{aligned} A_0(N \otimes M) &\simeq (2\varepsilon_1\varepsilon_2, d) \otimes (-1,-1)^{\binom{n+1}{4}} \otimes (1,1)^2 \\ &\quad \otimes \mathbf{Q}\left(\sqrt{-\varepsilon_2(-1)^{\binom{n+1}{2}}}\right). \end{aligned}$$

It then follows from Theorem 2.6.2 and Proposition 2.5.1 that

$$A^{\lambda,\mu}(\alpha) \simeq (-1,-1)^{\binom{n+1}{4}} (2\varepsilon_1\varepsilon_2, Z_\lambda^*)$$

in $Br\left(\mathbf{Q}(\chi^{\lambda,\mu})\right)$ with $\mathbf{Q}(\chi^{\lambda,\mu}) = \mathbf{Q}\left(\sqrt{-\varepsilon_2(-1)^{\binom{n+1}{2}}}\right)$.

If $2|n$, by Lemma 3.8.7 we get

$$A_0(N \otimes M) \simeq (2\varepsilon_1\varepsilon_2, d) \otimes (-1,-1)^{\binom{n+1}{4}} \otimes (1,1)^2 \otimes \left(\varepsilon_2, (-1)^{n/2}\right).$$

It then follows from Theorem 2.6.2 and Proposition 2.5.1 that

$$A^{\lambda,\mu}(\alpha) = (-1,-1)^{\binom{n+1}{4}} \left(\varepsilon_2, (-1)^{n/2}\right) (2\varepsilon_1\varepsilon_2, Z_\lambda^*)$$

in $Br\left(\mathbf{Q}(\chi^{\lambda,\mu})\right)$ with $\mathbf{Q}(\chi^{\lambda,\mu}) = \mathbf{Q}$.

The proof of the theorem is now complete. ∎

4.7 $[\varepsilon_1, \varepsilon_2, +1, -1, -1]$

With the knowledge of $\Psi^{\varepsilon_1,\varepsilon_2}$ and $\Phi^{\lambda,\mu}$, we are now able to characterize all spin characters of the double covers $W_n[\varepsilon_1, \varepsilon_2, 1, -1, -1]$. The following theorem gives us the structures of spin characters of $W_n[\varepsilon_1, \varepsilon_2, +1, -1, -1]$ in terms of $\Psi^{\varepsilon_1,\varepsilon_2}$ and $\Phi^{\lambda,\mu}$.

Theorem 4.7.1 *For even n, the irreducible spin representations of $W_n[\varepsilon_1, \varepsilon_2, +1, -1, -1]$ are $\psi^{\varepsilon_1,\varepsilon_2}\varphi^{\lambda,\emptyset}$ ($\lambda \in DP^+$) and $\psi^{\varepsilon_1,\varepsilon_2}\varphi^{\lambda,\emptyset}_{\pm}$ ($\lambda \in DP^-$). For odd n, the irreducible spin representations of $W_n[\varepsilon_1, \varepsilon_2, +1, -1, -1]$ are $\psi^{\varepsilon_1,\varepsilon_2}_{\pm}\varphi^{\lambda,\emptyset}$ ($\lambda \in DP^+$) and $\psi^{\varepsilon_1,\varepsilon_2}_{\pm}\varphi^{\lambda,\emptyset}_{\pm}$ ($\lambda \in DP^-$), where λ ranges over all partitions of n.*

Proof: The case $\varepsilon_1 = 1 = \varepsilon_2$ follows readily from Theorem 9.3 of Stembridge [23]. The remaining cases follow from the same proof of Theorem 9.3 of [23]. ∎

Theorem 4.7.2 *Let $\alpha = [\varepsilon_1, \varepsilon_2, +1, -1, -1]$, χ^λ be any character indexed by λ in the above theorem, and let $A^\lambda(\alpha)$, as an element of $Br(\mathbf{Q}(\chi^{\lambda,\mu}))$, be the central simple algebra associated with the character χ^λ. Recall that Definition 3.5.17 gives us $\mathcal{D}_0^{\lambda,\mu}, \mathcal{D}_1^{\lambda,\mu}, \mathcal{D}_2^{\lambda,\mu}, d_0^{\lambda,\mu}, d_1^{\lambda,\mu}$ and $d_2^{\lambda,\mu}$. Then*

(1) If $\lambda \in DP^+$, and $2 \not| n$,then

$$A^{\lambda,\mu}(\alpha) = (-1,-1)^{\binom{n+1}{4}}(2\varepsilon_1\varepsilon_2, d_1^{\lambda,\emptyset})\mathcal{D}_1^{\lambda,\emptyset},$$

with $\mathbf{Q}(\chi^\lambda) = \mathbf{Q}\left(\sqrt{-\varepsilon_2(-1)^{\binom{n+1}{2}}}\right)$.

(2) If $\lambda \in DP^+$ and $2|n$, then

$$A^\lambda(\alpha) = (-1,-1)^{\binom{n+1}{4}}\left(\varepsilon_2, (-1)^{n/2}\right)(2\varepsilon_1\varepsilon_2, d_1^{\lambda,\emptyset})\mathcal{D}_1^{\lambda,\emptyset},$$

with $\mathbf{Q}(\chi^\lambda) = \mathbf{Q}$.

(3) If $\lambda \in DP^-$, and $2 \not| n$, then

$$A^{\lambda,\mu}(\alpha) \simeq (-1,-1)^{\binom{n+1}{4}}\left(-\varepsilon_2(-1)^{\binom{n+1}{2}}, -2\varepsilon_1\varepsilon_2 d_0^{\lambda,\emptyset}\right)\mathcal{D}_0^{\lambda,\emptyset},$$

with $\mathbf{Q}(\chi^\lambda) = \mathbf{Q}$.

(4) if $\lambda \in DP^-$*, and* $2|n$*, then*

$$A^\lambda(\alpha) = (-1,-1)^{\binom{n+1}{4}}(\varepsilon_2,(-1)^{n/2})\mathcal{D}_0^{\lambda,\emptyset},$$

with $\mathbf{Q}(\chi^\lambda) = \mathbf{Q}\left(\sqrt{-2\varepsilon_1\varepsilon_2 d_0^{\lambda,\emptyset}(-1)^{n/2}}\right)$.

Proof: Let $G = W_n[\varepsilon_1,\varepsilon_2,1,-1,-1]$. Let N be a $\mathbf{Q}W_n[+1,+1,-1,+1,+1]$-module affording $r\varphi^{\lambda,\emptyset}$ if $\lambda \in DP^+$, $r(\varphi^{\lambda,\emptyset} + \varepsilon\varphi^{\lambda,\emptyset})$ if $\lambda \in DP^-$ and let M be a $\mathbf{Q}W_n[\varepsilon_1,\varepsilon_2,-1,\ -1,-1]$-module affording $\Im$ defined in Section 3.6. Then $W = N \otimes M$, as a $\mathbf{Q}W_n[\varepsilon_1,\varepsilon_2,+1,-1,-1]$-module affords the twisted product

$$\begin{aligned} r\varphi^{\lambda,\emptyset}\Im &\quad \text{if } \lambda \in DP^+ \\ r(\varphi_+^{\lambda,\emptyset} + \varphi_-^{\lambda,\emptyset})\Im &\quad \text{if } \lambda \in DP^-. \end{aligned}$$

We want to calculate $A_0(W) := A_0(W)$.

First, we consider the case $\lambda \in DP^+$. We have $\varepsilon\varphi^{\lambda,\emptyset} = \varphi^{\lambda,\emptyset}$ *and* $\mathbf{Q}\left(\varphi^{\lambda,\emptyset}\right) = \mathbf{Q}$.

Suppose $2 \not| n$. By Corollary 3.6.11, we have

$$\Im = 2(\psi + \varepsilon\psi + \delta\psi + \varepsilon\delta\psi) \text{ where } \psi = \psi^{\varepsilon_1,\varepsilon_2},$$

this implies W affords

$$\begin{aligned} \Im r\varphi^{\lambda,\emptyset} &= 2r(\psi\varphi^{\lambda,\emptyset} + \varepsilon\psi\varphi^{\lambda,\emptyset} + \delta\psi\varphi^{\lambda,\emptyset} + \varepsilon\delta\psi\varphi^{\lambda,\emptyset}) \\ &= 4r(\psi\varphi^{\lambda,\emptyset} + \delta\psi\varphi^{\lambda,\emptyset}). \end{aligned}$$

It follows that

$$dim_{\mathbf{Q}} A_0(W) = 4^2r^22 = 4^22r^2.$$

Next suppose $2|n$. We have $\Im = 4(\psi + \varepsilon\psi)$, $W = M \otimes N$ affords the twisted product

$$\begin{aligned} \Im(r\varphi^{\lambda,\emptyset}) &= 4r(\psi\varphi^{\lambda,\emptyset} + \varepsilon\psi\varphi^{\lambda,\emptyset}) \\ &= 8r(\psi\varphi^{\lambda,\emptyset}), \end{aligned}$$

this implies

$$dim_{\mathbf{Q}} A_0(W) = 8^2 r^2 = 4^3 r^2.$$

On the other hand, by Corollary 3.6.12 we have

$$dim_{\mathbf{Q}} A_0(M) = \begin{cases} 16 & \text{if } 2 \not| n \\ 32 & \text{if } 2|n. \end{cases}$$

Notice also $dim_{\mathbf{Q}} A_0(N) = r^2$, $L(N) = \{1, \varepsilon\}$ and $L(M) = \{1, \varepsilon, \delta, \varepsilon\delta\}$. Then it is straightforward to check

$$dim_{\mathbf{Q}} A_0(W) = dim_{\mathbf{Q}} A_0(N) \times dim_{\mathbf{Q}} A_0(M) \times |L(N) \cap L(M)|.$$

Consider the group pair $(W_n[+1, +1, -1, +1, +1], H)$, where H is the kernel of ε. By Corollary 3.5.8, $A := End_{\mathbf{Q}H}(M)$ is a **CSGA** of odd type. Let

$$\begin{aligned} A &\simeq [\mathcal{D}_1^{\lambda,\emptyset}, 1, d_1^{\lambda,\emptyset}], A_0 = A_0(N) \\ Z(A_0) &\simeq \mathbf{Q} + i\mathbf{Q}, i^2 = d_1^{\lambda,\emptyset} \in \mathbf{Q}^\times, \end{aligned}$$

where i can be chosen to be an ε-associate.

With the notations $S_M = S_M^\varepsilon, J_M$ of Definition 3.8.5, by Corollary 3.7.5 we have

$$\begin{aligned} A_0(N \otimes M) &\simeq A_0(N) \otimes (J_M^2, S_N^2 S_M^2) \otimes C_{A_0(M)}(S_M) \\ &\simeq A_0(N) \otimes (2\varepsilon_1\varepsilon_2, d_1^{\lambda,\emptyset}) \otimes C_{A_0(M)}(S_M). \end{aligned}$$

If $2 \not| n$, it follows from Lemma 3.8.8

$$\begin{aligned} A_0(W) &\simeq A_0(N) \otimes (-1,-1)^{\binom{n+1}{2}} \otimes (1,1)^2 \otimes (2\varepsilon_1\varepsilon_2, d_1^{\lambda,\emptyset}) \\ &\otimes \mathbf{Q}\left(\sqrt{-\varepsilon_2(-1)^{\binom{n+1}{2}}}\right). \end{aligned}$$

It then follows from Theorem 2.6.2 that

$$A^{\lambda,\mu}(\alpha) = (-1,-1)^{\binom{n+1}{2}} (2\varepsilon_1\varepsilon_2, d_1^{\lambda,\emptyset}) \mathcal{D}_1^{\lambda,\emptyset}$$

in $Br\left(\mathbf{Q}(\chi^{\lambda,\emptyset})\right)$ with $\mathbf{Q}(\chi^{\lambda,\mu}) = \mathbf{Q}\left(\sqrt{-\varepsilon_2(-1)^{\binom{n+1}{2}}}\right)$.

Next if $2|n$, it then follows from Lemma 3.8.8 that

$$\begin{aligned} A_0(W) &\simeq A_0(N) \otimes (-1,-1)^{\binom{n+1}{4}} \otimes (1,1)^2 \otimes (2\varepsilon_1\varepsilon_2, d_1^{\lambda,\emptyset}) \\ &\quad \otimes \left(\varepsilon_2, (-1)^{n/2}\right). \end{aligned}$$

It then follows from Theorem 2.6.2 that

$$A^{\lambda,\mu}(\alpha) = (-1,-1)^{\binom{n+1}{4}} \left(\varepsilon_2, (-1)^{n/2}\right) (2\varepsilon_1\varepsilon_2, d_1^{\lambda,\emptyset}) \mathcal{D}_1^{\lambda,\emptyset}$$

in $Br\left(\mathbf{Q}(\chi^{\lambda,\mu})\right)$ with $\mathbf{Q}(\chi^{\lambda,\mu}) = \mathbf{Q}$.

Next consider the case $\lambda \in DP^-$. We have $\varepsilon\varphi_+^{\lambda,\emptyset} \neq \varphi_+^{\lambda,\emptyset}$.

Suppose $2 \not| n$. By Corollary 3.6.11, we have

$$\Im = 2(\psi + \varepsilon\psi + \delta\psi + \varepsilon\delta\psi)$$

where $\psi = \psi^{\varepsilon_1,\varepsilon_2}$. This implies W affords

$$\Im r(\varphi^{\lambda,\emptyset} + \varepsilon\varphi^{\lambda,\emptyset}) = 4r(\psi\varphi^{\lambda,\emptyset} + \varepsilon\psi\varphi^{\lambda,\emptyset} + \delta\psi\varphi^{\lambda,\emptyset} + \varepsilon\delta\psi\varphi^{\lambda,\emptyset}).$$

It follows that

$$dim_{\mathbf{Q}} A_0(W) = 4^2 r^2 4 = 4^3 r^2.$$

Suppose next $2|n$. We have $\Im = 4(\psi + \varepsilon\psi)$, $W = M \otimes N$ affords the twisted product $\Im(r\varphi^{\lambda,\emptyset})$ and

$$\Im r(\varphi^{\lambda,\emptyset} + \varepsilon\delta\psi) = 8r(\psi\varphi^{\lambda,\emptyset} + \varepsilon\psi\varphi^{\lambda,\emptyset}),$$

this implies

$$dim_{\mathbf{Q}} A_0(W) = 8^2 r^2 2 = 4^3 r^2 2.$$

On the other hand, by Corollary 3.6.12 we have

$$dim_{\mathbf{Q}} A_0(M) = \begin{cases} 16 & \text{if } 2 \not| n \\ 32 & \text{if } 2|n. \end{cases}$$

Notice also $dim_{\mathbf{Q}} A_0(N) = 2r^2$, $L(N) = \{1, \varepsilon\}$ and $L(M) = \{1, \varepsilon, \delta, \varepsilon\delta\}$. Then it is straightforward to check

$$dim_{\mathbf{Q}} A_0(W) = dim_{\mathbf{Q}} A_0(N) \times dim_{\mathbf{Q}} A_0(M) \times |L(N) \cap L(M)|.$$

Consider the group pair $(W_n[+1, +1, -1, +1, +1], H)$, where H is the kernel of ε. By Corollary 3.5.8, $A := End_{\mathbf{Q}H}(M)$ is a **CSGA** of even type. Let

$$\begin{aligned} A &\simeq [\mathcal{D}_0^{\lambda,\emptyset}, 0, d_0^{\lambda,\emptyset}], A_0 = A_0(N), \\ Z(A_0) &\simeq \mathbf{Q} + \mathbf{Q}i, i^2 = d_0^{\lambda,\emptyset} \in \mathbf{Q}^\times. \end{aligned}$$

By Corollary 3.5.15, $\mathcal{D}_0^{\lambda,\emptyset}$ can be chosen to be a quaternion algebra. It is easy to check that $dim_{\mathbf{Q}} A = 2 \times dim_{\mathbf{Q}} A_0$.

with the notations $S_M = S_M^\varepsilon, J_M$ of Definition 3.8.5, by Corollary 3.7.5, we have

$$\begin{aligned} A_0(W) &\simeq \; < A_0(N) \otimes 1, S_N \otimes S_M > \otimes < 1 \otimes C_{A_0(M)}(S_M), i \otimes J_M > \\ &\simeq \; A_H(N) \otimes < 1 \otimes C_{A_0(M)}(S_M), i \otimes J_M > \end{aligned}$$

since $< A_0(N) \otimes 1, S_N \otimes S_M > \simeq A_0(N)$ by Corollary 2.5.7.

Suppose $2 \not| n$. By Lemma 3.8.9 we get

$$A_0(W) \;\simeq\; A_H(N) \otimes (-1, -1)^{\binom{n+1}{4}} \otimes (1, 1)^2 \otimes \left(-\varepsilon_2(-1)^{\binom{n+1}{2}}, -2\varepsilon_1\varepsilon_2 d_0^{\lambda,\emptyset}\right).$$

It then follows from Theorem 2.6.2 that

$$A^{\lambda,\mu}(\alpha) = (-1, -1)^{\binom{n+1}{4}} \left(-\varepsilon_2(-1)^{\binom{n+1}{2}}, -2\varepsilon_1\varepsilon_2 d_0^{\lambda,\emptyset}\right) \mathcal{D}_0^{\lambda,\emptyset}$$

in $Br\left(\mathbf{Q}(\chi^{\lambda,\mu})\right)$ with $\mathbf{Q}(\chi^{\lambda,\mu}) = \mathbf{Q}$.

Next suppose $2|n$. By lemma 3.8.9 we get

$$\begin{aligned} A_0(W) \;\simeq\; & A_H(N) \otimes (-1, -1)^{\binom{n+1}{4}} \otimes (1, 1)^2 \otimes \left(\varepsilon_2, (-1)^{n/2}\right) \\ & \otimes \mathbf{Q}\left(\sqrt{-2\varepsilon_1\varepsilon_2 d_0^{\lambda,\emptyset}(-1)^{n/2}}\right). \end{aligned}$$

It then follows from Theorem 2.6.2

$$A^{\lambda,\mu}(\alpha) = (-1,-1)^{\binom{n+1}{4}} \left(\varepsilon_2, (-1)^{n/2}\right) \mathcal{D}_0^{\lambda,\emptyset}$$

in $Br\left(\mathbf{Q}(\chi^{\lambda,\mu})\right)$ with $\mathbf{Q}(\chi^{\lambda,\mu}) = \mathbf{Q}\left(\sqrt{-2\varepsilon_1\varepsilon_2 d_0^{\lambda,\emptyset}(-1)^{n/2}}\right)$.

Now the proof of the theorem is complete. ■

4.8 $[\varepsilon_1, \varepsilon_2, +1, +1, -1]$

According to Theorem 2.4.3, we can characterize all spin representations of the double cover $W_n[\varepsilon_1, \varepsilon_2, 1, +1, -1]$ in terms of Θ_0, $\Psi^{\varepsilon_1,\varepsilon_2}$, and $\Phi^{\lambda,\mu}$. We have the following theorem:

Theorem 4.8.1 *The irreducible spin representations of $W_n[\varepsilon_1, \varepsilon_2, +1, +1, -1]$ can be labeled by the partitions λ of n in DP, so that λ indexes submodules of $\Theta_0\Psi^{\varepsilon_1,\varepsilon_2}\Phi^{\lambda,\emptyset}$. In the following table, n_λ denotes the number of modules indexed by λ, m_λ denotes their multiplicity in $\Theta_0\Psi^{\varepsilon_1,\varepsilon_2}\Phi^{\lambda,\emptyset}$, and o_λ denotes the size of the orbit of λ.*

n_λ	o_λ	m_λ	
1	1	2	if $\lambda \in DP^+$, and n is even
2	2	1	if $\lambda \in DP^+$, and n is odd or $\lambda \in DP^-$ and n is even
1	4	1	if $\lambda \in DP^-$, and n is odd .

Proof: By Proposition 3.3.3 we have $\Theta^{\varepsilon_1,\varepsilon_2} = 2\Theta_0^{\varepsilon_1,\varepsilon_2}$. Then the case $\varepsilon_1 = 1 = \varepsilon_2$ follows readily from Corollary 9.5 of Stembridge [23]. The remaining cases follow from Stembridge's argument, using Theorem 3.3.5 and Theorem 2.4.3. ■

Theorem 4.8.2 *Let $\alpha = [\varepsilon_1, \varepsilon_2, +1, +1, -1]$, let χ^λ be any character indexed by λ in the above theorem, and let $A^\lambda(\alpha)$, as an element of $Br(\mathbf{Q}(\chi^{\lambda,\mu}))$, be the central simple algebra associated with the character χ^λ. Recall that Definition 3.5.17 gives us $\mathcal{D}_0^{\lambda,\mu}$, $\mathcal{D}_1^{\lambda,\mu}$, $\mathcal{D}_2^{\lambda,\mu}$, $d_0^{\lambda,\mu}$, $d_1^{\lambda,\mu}$ and $d_2^{\lambda,\mu}$. Then*

(1). If $\lambda \in DP^+$, and $2 \not| n$, *then*

$$A^\lambda(\alpha) \simeq (-1,-1)^{\binom{n+1}{4}}(2\varepsilon_1\varepsilon_2, d_1^{\lambda,\emptyset})\mathcal{D}_1^{\lambda,\emptyset},$$

with $\mathbf{Q}(\chi^\lambda) = \mathbf{Q}\left(\sqrt{-\varepsilon_2(-1)^{\binom{n+1}{2}}}\right)$.

(2) If $\lambda \in DP^+$, and $2|n$, *then*

$$A^\lambda(\alpha) \simeq (-1,-1)^{\binom{n+1}{4}}(2\varepsilon_1\varepsilon_2, d_1^{\lambda,\emptyset})\left((-1)^{n/2},\varepsilon_2\right)\mathcal{D}_1^{\lambda,\emptyset},$$

with $\mathbf{Q}(\chi^\lambda) = \mathbf{Q}$.

(3) If $\lambda \in DP^-$, and $2 \not| n$, *then*

$$A^\lambda(\alpha) = (-1,-1)^{\binom{n+1}{4}}\left(-\varepsilon_2(-1)^{\binom{n+1}{2}}, -2\varepsilon_1\varepsilon_2 d_0^{\lambda,\emptyset}\right)\mathcal{D}_0^{\lambda,\emptyset},$$

with $\mathbf{Q}(\chi^\lambda) = \mathbf{Q}$.

(4) If $\lambda \in DP^-$, and $2|n$, *then*

$$A^\lambda(\alpha) = (-1,-1)^{\binom{n+1}{4}}\left(\varepsilon_2,(-1)^{n/2}\right)\mathcal{D}_0^{\lambda,\emptyset},$$

with $\mathbf{Q}(\chi^\lambda) = \mathbf{Q}\left(\sqrt{-2\varepsilon_1\varepsilon_2 d_0^{\lambda,\emptyset}(-1)^{n/2}}\right)$.

Proof: Let $G = W_n[\varepsilon_1,\varepsilon_2,+1,+1,-1]$. Let $m_{\mathbf{Q}}(\varphi^{\lambda,\emptyset}) = r$, then by Corollary 3.5.16, $r = 1$ or 2. Let N be a $\mathbf{Q}W_n[+1,+1,+1,-1,-1]$-module affording $r\varphi^{\lambda,\emptyset}$ if $2 \not| n$, $r(\varphi^{\lambda,\emptyset} + \varepsilon\varphi^{\lambda,\emptyset})$ if $2|n$, let O be a $\mathbf{Q}W_n[+1,+1,+1,+1,+1]$-module affording θ_0, and let P a $\mathbf{Q}W_n[\varepsilon_1,\varepsilon_2,-1,-1,-1]$-module affording $\Im$ defined in Section 3.6. Let $M = O \otimes P$ be the twisted product of O and P. Then $W = N \otimes M$, as a $\mathbf{Q}W_n[\varepsilon_1,\varepsilon_2,+1,+1,-1]$-module , affords the twisted product

$$(r\varphi^{\lambda,\emptyset})\theta_0\Im \quad \text{if } 2 \not| n$$
$$r(\varphi^{\lambda,\emptyset} + \varepsilon\varphi^{\lambda,\emptyset})\theta_0\Im \quad \text{if } 2|n.$$

We want to calculate $A_0(W) := End\mathbf{Q}G(W)$.

By Lemma 3.8.4.

$$dim_{\mathbf{Q}}A_0(W) = dim_{\mathbf{Q}}A_0(N) \times dim_{\mathbf{Q}}A_0(M) \times |L(N) \cap L(M)|,$$

and

$$dim_{\mathbf{Q}}A_0(M) = dim_{\mathbf{Q}}A_0(O) \times dim_{\mathbf{Q}}A_0(P) \times |L(O) \cap L(P)|.$$

So whenever we apply the theorems of Section 3.7, we do not need to worry about checking the dimesions.

We divide the proof into a number of cases.

Let's first consider $\lambda \in DP^+$. We have $\varepsilon\varphi^{\lambda,\emptyset} = \varphi^{\lambda,\emptyset}$ *and* $\mathbf{Q}(\varphi^{\lambda,\emptyset}) = \mathbf{Q}$.

Consider the group pair $(W_n[W_n[+1,+1,+1,-1,-1], H)$, where H is the kernel of ε. By Corollary 3.5.8, $A := End_{\mathbf{Q}H}(N)$ is a **CSGA** of odd type. Let

$$\begin{aligned} A &\simeq [\mathcal{D}_1^{\lambda,\emptyset}, 1, d_1^{\lambda,\emptyset}],\ A_0 = A_0(N) \\ Z(A_0) &\simeq \mathbf{Q} + S_N\mathbf{Q},\ S_N^2 = d_1^{\lambda,\emptyset} \in \mathbf{Q}^\times. \end{aligned}$$

With the notations $S_N, S_M = S_M^\varepsilon, J_M$ of Definition 3.8.5, by Corollary 3.7.5 we have

$$\begin{aligned} A_0(N \otimes M) &\simeq A_0(N) \otimes (J_M^2, S_N^2 S_M^2) \otimes C_{A_0(M)}(S_M) \\ &\simeq A_0(N) \otimes (J_M^2, dS_M^2) \otimes C_{A_0(M)}(S_M) \\ &\simeq A_0(N) \otimes (2\varepsilon_1\varepsilon_2, d) \otimes C_{A_0(M)}(S_M). \end{aligned}$$

To be specific, we distinguish the two cases $2|n$ and $2 \not| n$.

If $2 \not| n$, by Lemma 3.8.7 we get

$$\begin{aligned} A_0(N \otimes M) \simeq\ & A_0(N) \otimes (2\varepsilon_1\varepsilon_2, d_1^{\lambda,\emptyset}) \otimes (-1,-1)^{\binom{n+1}{4}} \otimes (1,1)^2 \\ & \otimes \mathbf{Q}\left(\sqrt{-\varepsilon_2(-1)^{\binom{n+1}{2}}}\right). \end{aligned}$$

It then follows from Theorem 2.6.2

$$A^{\lambda,\mu}(\alpha) = (-1,-1)^{\binom{n+1}{4}} (2\varepsilon_1\varepsilon_2, d_1^{\lambda,\emptyset})\mathcal{D}_1^{\lambda,\emptyset}$$

in $Br\left(\mathbf{Q}(\chi^{\lambda,\mu})\right)$ with $\mathbf{Q}(\chi^{\lambda,\mu}) = \mathbf{Q}\left(\sqrt{-\varepsilon_2(-1)^{\binom{n+1}{2}}}\right)$.

If $2|n$, by Lemma 3.8.7 we get

$$\begin{aligned} A_0(N \otimes M) \simeq\ & A_0(N) \otimes (2\varepsilon_1\varepsilon_2, d_1^{\lambda,\emptyset}) \otimes (-1,-1)^{\binom{n+1}{4}} \otimes (1,1)^2 \\ & \otimes \left(\varepsilon_2, (-1)^{n/2}\right). \end{aligned}$$

It then follows from Theorem 2.6.2 and proposition 2.5.1

$$A^{\lambda,\mu}(\alpha) \simeq (-1,-1)^{\binom{n+1}{4}}(2\varepsilon_1\varepsilon_2, d_1^{\lambda,\emptyset}) \left(\varepsilon_2, (-1)^{n/2}\right) \mathcal{D}_1^{\lambda,\emptyset}$$

in $Br\left(\mathbf{Q}(\chi^{\lambda,\mu})\right)$ with $\mathbf{Q}(\chi^{\lambda,\mu}) = \mathbf{Q}$.

Next we consider $\lambda \in DP^-$. We have $L(N) = \{1, \varepsilon\}$.

Consider the group pair $(W_n[W_n[+1,+1,+1,-1,-1], H)$, where H is the kernel of ε. By Corollary 3.5.8, $A := End_{\mathbf{Q}H}(N)$ is a **CSGA** of even type. Let

$$\begin{aligned} A &\simeq [\mathcal{D}_0^{\lambda,\emptyset}, 0, d_0^{\lambda,\emptyset}], A_0 = A_0(N), \\ Z(A_0) &\simeq \mathbf{Q} + i\mathbf{Q},\ i^2 = d_0^{\lambda,\emptyset} \in \mathbf{Q}^\times. \end{aligned}$$

It is easy to check $dim_{\mathbf{Q}}A = 2 \times dim_{\mathbf{Q}}A_0$. By Corollary 3.5.15 $\mathcal{D}_0^{\lambda,\emptyset}$ can be chosen to be a quaternion algebra.

With the notations $S_N, S_M = S_M^\varepsilon, J_M$ of Definition 3.8.5, by Theorem 3.7.4 we have

$$\begin{aligned} A_0(N \otimes M) &\simeq\ < A_0(N) \otimes S_N \otimes S_M > \otimes < 1 \otimes C_{A_0(M)}(S_M), i \otimes J_M > \\ &\simeq\ A \otimes < 1 \otimes C_{A_0(M)}(S_M), i \otimes J_M > \end{aligned}$$

since $< A_0(N) \otimes S_N \otimes S_M > \simeq A$, this follows from the fact $S_M^2 = 1$ and Theorem 2.5.6.

If $2 \not| n$, by Lemma 3.8.9 we have

$$A_0(W) \simeq A \otimes (-1,-1)^{\binom{n+1}{4}} \otimes (1,1)^2 \otimes \left(-\varepsilon_2(-1)^{\binom{n+1}{2}}, -2\varepsilon_1\varepsilon_2 d_0^{\lambda,\emptyset}\right).$$

It then follows from Theorem 2.6.2 that

$$A^{\lambda,\mu}(\alpha) \simeq (-1,-1)^{\binom{n+1}{4}} \left(-\varepsilon_2(-1)^{\binom{n+1}{2}}, -2\varepsilon_1\varepsilon_2 d_0^{\lambda,\emptyset}\right) \mathcal{D}_0^{\lambda,\emptyset}$$

in $Br\left(\mathbf{Q}(\chi^{\lambda,\mu})\right)$ with $\mathbf{Q}(\chi^{\lambda,\mu}) = \mathbf{Q}$.

If next $2|n$, by Lemma 3.8.9 we have

$$\begin{aligned} A_0(W) \simeq\ & A \otimes (-1,-1)^{\binom{n+1}{4}} \otimes \left(\varepsilon_2, (-1)^{n/2}\right) \otimes (1,1)^2 \\ & \otimes \mathbf{Q}\left(\sqrt{-2\varepsilon_1\varepsilon_2 d_0^{\lambda,\emptyset}(-1)^{n/2}}\right). \end{aligned}$$

It then follows from Theorem 2.6.2 that

$$A^{\lambda,\mu}(\alpha) \simeq (-1,-1)^{\binom{n+1}{4}} \left(\varepsilon_2, (-1)^{n/2}\right) \mathcal{D}_0^{\lambda,\emptyset}$$

in $Br\left(\mathbf{Q}(\chi^{\lambda,\mu})\right)$ with $\mathbf{Q}(\chi^{\lambda,\mu}) = \mathbf{Q}\left(\sqrt{-2\varepsilon_1\varepsilon_2 d_0^{\lambda,\emptyset}(-1)^{n/2}}\right)$.

This completes the proof of the theorem. ∎

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Zhaowei Du was born at Tianjin of P. R. China, on December 4, 1958. In February 1982 he received his B.S. (Math) from Hebei University. Shortly after that he entered Wuhan University and received his M.S. (Math) in July, 1986. Then he was hired by Tsinghua University in July, 1986 and began his six years teaching career as an assistant professor of the Applied Mathematics Department until he left for the United States in 1992 to pursue his Ph.D. Since August 1992, he has been studying at the University of Florida. In the summer of 1994, he received his second M.S. (Math) from the University of Florida. In 1996, he entered a concurrent program of computer science and received a M.S. from the Computer Information Science and Engineering (CISE) Department in December, 1997. Now he is devoting himself to mathematics and is planning to complete his Ph.D. dream in Spring, 1998.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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